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THE PRINCIPLES OF MECHANICS

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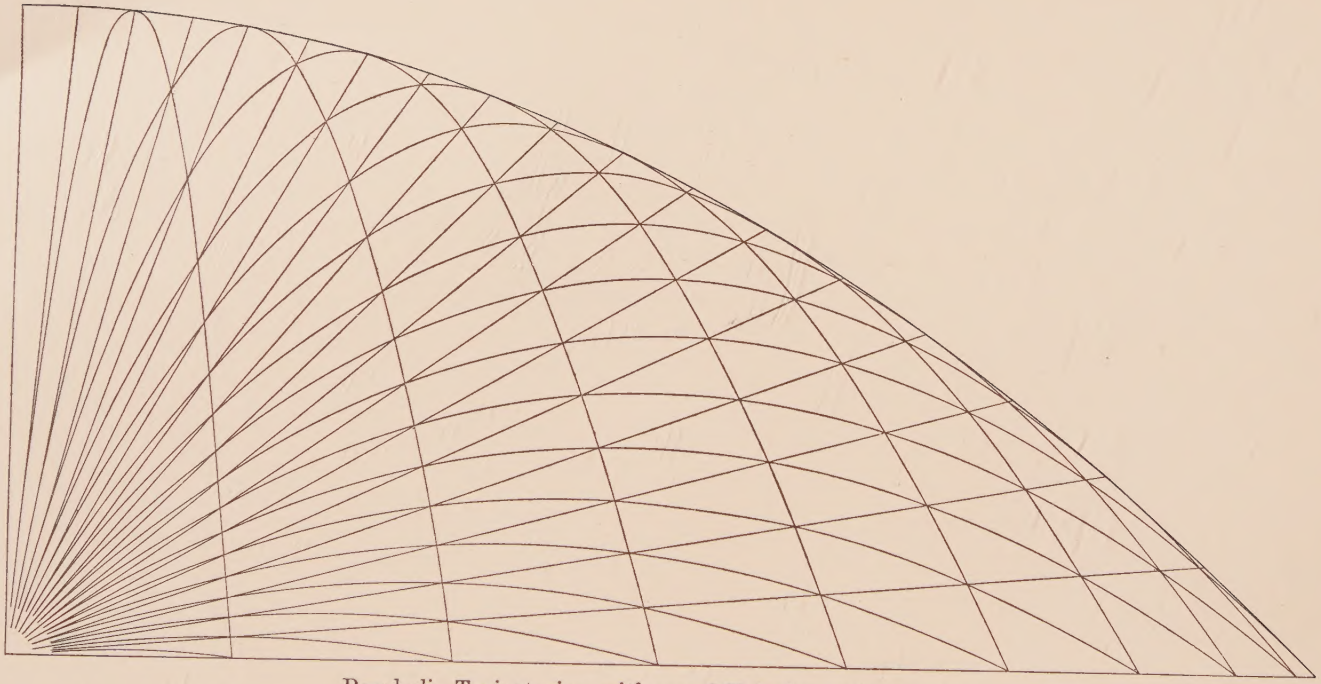

Parabolic Trajectories, with equal Velocity of Projection.

THE PRINCIPLES OF MECHANICS

AN ELEMENTARY COURSE

BY

H. C. PLUMMER

M.A., F.R.S.

LATE ROYAL ASTRONOMER OF IRELAND

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PREFACE

The study of mechanics falls quite naturally into two stages. The first can be embarked upon profitably with only a slight equipment of mathematical knowledge. Before the second can be undertaken it is necessary to have gained some familiarity with geometry and analysis, and particularly with differential equations.

This book is concerned only with the first stage. But there are two possible ways of treating it. In the one a certain range of elementary problems is contemplated, and these are provided with solutions appropriate to the particular case without much regard to the need for a coherent foundation. It is a method which can be justified in the case of those students who are not destined to pursue the subject further, and it has undeniable advantages when the object in view is merely to gain marks in examinations of a familiar type. But it is not suitable for those who will ultimately go further. For them a method which treats a small part of a subject as though it were a whole is apt to be nothing less than disastrous.

Accordingly, without trenching on the developments which belong properly to the second stage, an attempt has been made in this book to treat the first in a way more appropriate to its subordinate and introductory character. Now it is difficult to present the fundamental ideas of dynamics without involving the related notions of the infinitesimal calculus. At the same time the approach to the calculus will have been made needlessly hard if the opportunity has been missed of associating its concepts with the concrete problems of mechanics. These convictions have determined the form of the book. But the reader should be warned that the beginnings of the calculus will still lie before him. All that is attempted beyond the strict limits of the present subject is to prepare his mind for further development.

The examples have been selected, and in many cases devised, with much care. Few of them will be found to be of the numerical type. No want of appreciation for the value of arithmetical training is to

be inferred. On the contrary, the art of computation is far too important to be cultivated by casual application to examples such as these. They are designed to exercise the student in giving a mathematical form to mechanical ideas. It is not right that his attention should be distracted from the main purpose by difficulties of another order. Those who dissent from this view should have no difficulty in supplying the omission, but the subject of this book is Mechanics, not arithmetic.

Grateful thanks are due to my colleagues, Mr. F. R. W. Hunt, M.A., Mr. C. E. Wright, M.Sc., and Captain A. V. Kerrison, R.A., for help in preparing diagrams for reproduction, in proof-reading, and for critical suggestions. The work of the publishers in the production of the book will speak for itself.

H. C. PLUMMER.

Military College of Science,
Woolwich, *November*, 1928.

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PART I

KINEMATICS

CHAPTER I

SCOPE OF SUBJECT. GEOMETRICAL PRINCIPLES

1. The subject of mechanics may be taken to embrace the motions of material bodies and the conditions which produce and govern these motions. For convenience it must be limited and divided into several parts, and the following divisions are in natural order :

(1) *The geometry of displacements*, in which the geometrical relations between bodies before and after displacement are considered, without any attention to the element of time or the *rate* at which the displacements occur. This is simply geometry applied to a special class of problems.

Statements such as that a free projectile moves in a parabola, or that a solid body is revolving round a fixed axis, involve relations of this kind. Other illustrations will follow in this chapter.

(2) *Kinematics*. Here the displacements are correlated with the time, thus adding a new element to the three dimensions of space. Motions are investigated and completely defined, but no consideration is given to causes, except so far as these depend on purely geometrical connexions and are independent of the idea of force.

Thus, when a wheel is rolled along the ground at a uniform rate, the motion of a selected point on its rim presents a purely kinematical problem, and the position of the point at any time can be calculated from given initial conditions without any regard to the forces under which the wheel is rolling.

(3) *Dynamics*, which in its widest sense comprehends the whole subject. In this division the idea of *mass* is required and *forces* are introduced as the cause of motion.

(4) *Statics.* This division of the subject is concerned with the reduction of complicated systems of forces to their simplest terms, and deals with the particular case of equilibrium, in which the forces balance and no motion is produced by them.

This seems the most logical order of development. But the divisions are so intimately related that it may not always be necessary or convenient to adhere to it.

2. This classification indicates an order of development according to principles and methods. But the same principles can be applied to matter in all its various forms, solid, liquid and gaseous, by appropriate methods. It is therefore necessary to lay down other divisions of the subject according to the nature of the material substances under consideration. Here the subject of mechanics will be strictly confined to solid bodies, the allied treatment of matter in a fluid state being excluded as constituting the distinct field of Hydromechanics (hydrostatics, hydrodynamics). Further, the moving bodies to be contemplated here will be not only solid but for the most part absolutely rigid, or incapable of deformation. If flexible strings are introduced, they will be supposed inextensible when stretched. But the property of perfect rigidity is ideal and is possessed by no material substance in nature. Hence, although this property is so nearly approached that the behaviour of such ideal bodies can be studied with great profit, it has seemed well not to exclude entirely from an elementary work the consideration of bodies possessing imperfect rigidity. This constitutes the subject of *Elasticity*, in which the transmission of forces through actual solids and their effects (strains) are more particularly studied. An introductory chapter on this subject will be found at the end of this book.

3. The subject of mechanics is a branch of Physics, and as such is closely related to other branches. A body in motion possesses *energy* in its simplest and most palpable form. As the motion changes under the influence of force, the form of the energy changes and may be traced in the shape of mechanical *work*. In certain cases the changes are registered completely in the changing configuration of the system and are independent of its intermediate states. The relation between energy and work can then be regarded as purely mechanical. But when such actions as friction or impact occur between the several parts of a system, the transformations of

energy are no longer purely mechanical but partly into heat, which is molecular or radiant energy. To trace the changes of this kind is the special function of Thermodynamics, and without following this branch of physics such changes, from the purely mechanical point of view, can only be regarded as unexplained loss.

Similarly, mechanical energy is capable of transformation into other forms, such as electromagnetic energy. To trace such changes lies altogether outside the scope of mechanics, and belongs to a distinct branch of physics, Electro-dynamics. While, however, it is clear that mechanics must be kept separate from the other departments of physics, it is well to realize that the doctrine of energy pervades them all, and that this doctrine, which is the connecting link binding the whole, has its foundations in pure dynamics.

These remarks must suffice to indicate the nature of the subject of mechanics, its limitations, and something of its place in the general scheme of physics.

4. In the first place, we consider the displacement of a point along a straight line.

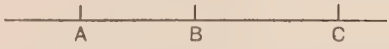

FIG. 1.

Let AB be a symbol denoting the displacement of a point from A to B. Then

$$AB+BC=AC \quad \text{.....(1)}$$

states a fact which will be readily accepted when A, B, C are in the order shown. But the mathematical form of the statement is useless unless it possesses a generality covering all cases and allowing the equation to be used in the ordinary way. Therefore, consider a different order of the points :

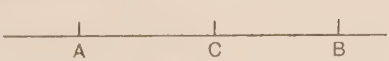

FIG. 2.

Here, in the primitive sense,

$$AC+CB=AB,$$

and according to the ordinary interpretation of an equation

$$AC=AB-CB. \quad \text{.....(2)}$$

By comparison, equations (1) and (2) are consistent, and (1) possesses the desired generality, provided $BC=-CB$. This is the ordinary rule of signs, and with it (1) gives an intelligible meaning in all

cases, no matter in what order the three points A, B, C occur. The symbol AB here denotes not merely the length between A and B, but that length associated with the direction *from A to B*, so that BA is similarly associated with the opposite direction from B to A.

5. But the equation (1) can be usefully accepted in a far more general sense. Thus, when A, B, C are no longer in a straight line,

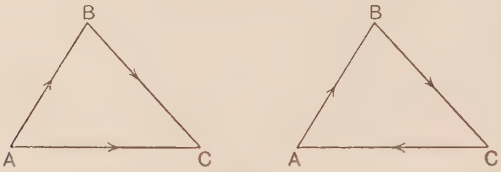

FIG. 3.

but form a triangle (which is necessarily plane), the equation

$$AB+BC=AC$$

may be read in the sense : the displacement of a point from A to B followed by its displacement from B to C is equivalent to its displacement from A to C. Here again the symbol AB must be understood as denoting a peculiar kind of quantity, a length associated with a certain direction. Were it merely a positive quantity in the ordinary algebraic sense the statement expressed by the equation would be palpably absurd, since the two sides of a triangle are together greater than the third. Reversing the direction of AC, we obtain

$$AB+BC+CA=0,$$

or in words, the result of successive displacements along the three sides of a triangle taken in order is null.

These statements constitute the theorem which may be called the *triangle of displacements*.

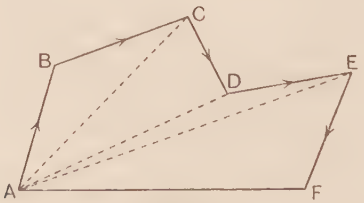

FIG. 4.

6. This theorem can be immediately and easily generalized. Let A, B, C, D, E, F be any points forming a polygon, with sides

taken in order. Then by repeated applications of the triangle of displacements,

$$\begin{aligned} AB+BC+CD+DE+EF &= AC+CD+DE+EF \\ &= AD+DE+EF \\ &= AE+EF \\ &= AF, \end{aligned}$$

or again,

$$AB+BC+CD+DE+EF+FA=0.$$

In words, the sum of the displacements along the sides of the polygon in order, returning to the starting point after a complete circuit, is null. Notice that there is nothing in the reasoning which requires the points of the polygon to be in a plane. They may be any points whatever, but the last result applies only to a *closed* polygon.

7. Consider next a parallelogram $ABCB'$, with diagonal AC.

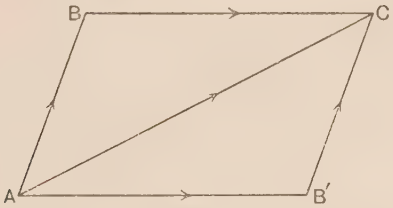

FIG. 5.

In this case, evidently,

$$AB+BC=AC=AB'+B'C.$$

Here AB is equal and parallel to $B'C$, and BC is equal and parallel to AB'. As defined by magnitude and direction the displacements AB, $B'C$ are the same, and so are BC, AB'. What this equation asserts is that the order in which the two displacements are made is immaterial to the result. Either one may follow the other, or indeed both may be considered to happen simultaneously. In this sense they are independent, and accordingly

$$AB+AB'=AB+BC=AC.$$

This expresses the theorem known as the *parallelogram of displacements*. When a point is subject to two independent displacements AB, AB', the *resultant* displacement is represented, in magnitude and direction, by the diagonal AC of the parallelogram of which AB, AB' are concurrent sides. Conversely, when AC is given together with any two directions AB, AB', by drawing CB, CB' parallel to each of these lines to cut the other, AC can be resolved

into its components AB, AB' along the given directions. From this it would be easy, in three dimensions, to proceed to a resolution of a given displacement in any three distinct directions not lying in one plane.

8. Quantities which are defined by a certain magnitude or intensity associated with a certain direction in space, and which can be resolved and combined according to the parallelogram law, are called *vectors*. Linear displacements of a point have been seen to obey this law and are one example out of many which will present themselves.

Not all magnitudes with an associated direction in space are vectors. Thus finite rotations about given axes are not vectors, and if a body is rotated about two axes in space through given angles successively, the final position is not independent of the order in which the rotations are effected. The property of combining according to the parallelogram law is an essential part of the definition of a vector.

From the analogy of any class of vector to a linear displacement it follows that any vector can be represented in a diagram by a straight line. The magnitude of the vector is represented by the length of the corresponding straight line on a certain arbitrary scale. If vectors of different kinds are represented in the same diagram, as often happens, it should be remembered that while each class must be represented on a consistent scale, the scales assigned to different classes are quite independent and may be chosen according to convenience.

9. It is frequently most convenient to resolve a vector along two

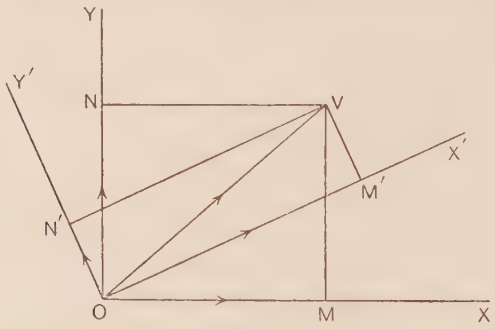

FIG. 6.

directions at right angles. Let OX, OY be axes drawn in such directions, and let $OV = r$ be a vector making the angle α with OX.

Let M and N be the feet of the perpendiculars on OX, OY. Then the components of r are

$$x = OM = r\cos\alpha,$$
$$y = ON = r\sin\alpha.$$

If the axes are turned through the angle $XOX' = \theta$ into the position OX', OY', and M', N' are the feet of the perpendiculars from V on OX', OY', the components of r become

$$x' = OM' = r\cos(\alpha - \theta),$$
$$y' = ON' = r\sin(\alpha - \theta),$$

since the angle $VOX' = \alpha - \theta$. Hence, without reference to the vector r, the components x', y' can be expressed in terms of x, y. For

$$\begin{aligned} x' &= r\cos\alpha\,.\cos\theta + r\sin\alpha\,.\sin\theta \\ &= \quad x\cos\theta + y\sin\theta, \\ y' &= r\sin\alpha\,.\cos\theta - r\cos\alpha\,.\sin\theta \\ &= \quad y\cos\theta - x\sin\theta. \end{aligned}$$

It is easier to write down these formulae when it is remembered that $x' = x$, $y' = y$ when $\theta = 0$, $\cos\theta = 1$, $\sin\theta = 0$, while $x' = y$, $y' = -x$ when $\theta = 90°$, $\cos\theta = 0$, $\sin\theta = 1$.

10. All natural bodies exist in three dimensions, and possess corresponding mobility, except in so far as this is reduced by limiting constraints. But we may begin by considering one and two dimensions.

The case of one dimension is represented by the displacement of a section AB of length b along a straight line. In order that a displacement of the section AB may be detected and measured, it

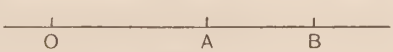

FIG. 7.

is necessary that a fixed point O shall be marked on the line. Then the length $OA = x$ can be measured, and its variation will determine the displacement of the section ; the displacement of any particular point in the section, such as B ($OB = x + b$), can be inferred. Thus one coordinate (as x) fixes the position of AB, and the section is said to have *one degree of freedom*. Notice that the order of the points A, B cannot be reversed : to do so would require a rotation, that is, a motion in two dimensions.

In two dimensions a plane area, sufficiently represented by a triangle ABC, moves in its own plane. The position of the area

can be fixed (*a*) by defining the line on which AB is situated, and (*b*) by defining the position of the section AB on this line. As fixed

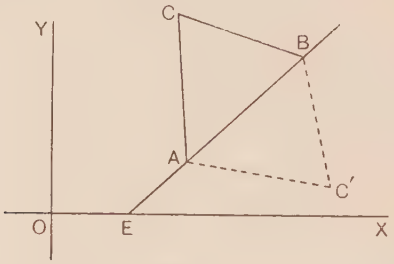

FIG. 8.

marks in the plane may be taken a fixed point O and a line OX drawn in a certain direction. To these may be added the perpendicular line OY, thus forming *a frame of reference*. The position of the line AB (unlimited) can then be determined by two quantities chosen according to convenience, such as the intercept $OE = a$ on OX and the angle $XEA = \theta$. Thus the line itself requires two independent quantities for its definition, and has *two* degrees of freedom. By this process it is also provided with a fixed point, such as E, and the segment AB can be located by a third quantity, to which corresponds a third degree of freedom. Therefore the area in two-dimensional motion has in all $2 + 1 = 3$ degrees of freedom.

If C' is the image of C in AB, the triangles ABC, ABC' are equal in all respects, but of opposite aspect. Notice that the triangle ABC, once laid down, cannot be brought into the position ABC' by displacement in the plane. This can only be effected by a rotation about AB, involving an excursion into the third dimension.

11. In three dimensions the body is fixed when three of its points ABC (not in a straight line) are fixed. To form a frame of reference it is necessary to have in space a fixed point O, a straight line OX and a certain plane containing OX. Then it is possible to draw a straight line OY perpendicular to OX in this plane, and a third line OZ perpendicular to the plane, making a frame of three axes OX, OY, OZ mutually at right angles. To locate the plane ABC now requires three independent quantities, such as the three intercepts on these three axes. The plane itself has therefore three degrees of freedom. At the same time, by its intersections with the axes, the plane is provided with points of reference, and the triangle ABC has three degrees of freedom and can be located in the plane when

the necessary three quantities are given. Hence in all the body in space of three dimensions possesses $3+3=6$ degrees of freedom, and requires 6 quantities or coordinates to define its position.

If P is any point connected with ABC, and P' is the image of P in the plane ABC, the pyramids $PABC$, $P'ABC$ are in every respect equal but opposite in aspect, bearing to one another the relation of a right and left hand, and no displacement can bring them into the same position.

To sum up, the free body in space of one dimension has 1 degree of freedom, in space of two dimensions $1+2=3$ degrees of freedom, and in space of three dimensions $1+2+3=6$ degrees of freedom; or in any case $\frac{1}{2}n(n+1)$ degrees, where n is the number of dimensions, 1, 2 or 3.

When the body is no longer free, the degrees of freedom are reduced by the number of independent conditions imposed. In particular, one is lost when a point of the body is caused to move on a fixed surface, two when it moves on a fixed curve (the intersection of two surfaces), and three when it is absolutely fixed in space. The constraints considered here must be independent. Thus, to fix two points of a body is not equivalent to exhausting six degrees of freedom, and thereby fixing the body; for obviously a rotation about the joining line is still possible. The reason is that, when the first point is fixed, the second is connected with it by its natural distance and enjoys only two remaining degrees of freedom, which must be supplemented by a third independent coordinate. The attempt to fix two points is bound to introduce strain, and should be avoided in the design of instruments. The true principles are well illustrated by the geometrical stand, wherein one foot rests in a cavity, a second in a groove, and a third freely on a horizontal surface.

12. The result of a displacement of a plane area in its own plane may be expressed as a simple rotation. As already seen, it is completely defined by a straight line AB displaced to the position $A'B'$. Let these lines intersect in C, and let O be the centre of the circle circumscribing the triangle $AA'C$. Let EOF be the diameter bisecting AA' at right angles at D. Then $EA=EA'$, and the angles EAB, $EA'B'$, being the supplements of the equal angles EAC, $EA'C$, are also equal. Hence a simple rotation round E through the angle AEA' will bring AB into coincidence with $A'B'$ and represent the given displacement.

Since $FA = FA'$ and the angles FAB, $FA'C$, being the supplements of the same angle CAF, are also equal, a rotation round F through the angle AFA' will bring AB into the position $A'B''$ opposite to $A'B'$, if this is required.

To this description of the displacement there is an apparent exception when AB, $A'B'$ are parallel. In this case C is at infinity, the circle drawn has an infinite radius and the arc AFA' coincides with the chord AA'. Then F coincides with D, the middle point of AA', and E is at infinity on the perpendicular bisector of AA'. Hence the displacement of AB to a parallel position $A'B'$, if in the same direction, may be regarded as a rotation about an infinitely distant point E through a vanishingly small angle AEA'. But this is more commonly and naturally described as a pure *translation*.

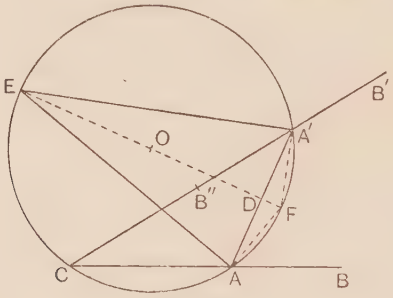

FIG. 9.

On the other hand, if AB is displaced to a parallel and opposite position $A'B''$, this is the effect of a rotation about D, the middle point of AA', through two right angles.

13. In regard to motion in a plane it is natural to speak of rotation about a point. In reality, a rotation always takes place about an axis, in this case a line through the point perpendicular (or normal) to the plane. The effect of rotations successively about two points, like any general displacement in the plane, must be equivalent to a single rotation, and the result next to be obtained for this effect will apply generally to successive rotations about two parallel axes.

Let the first rotation be through the angle α about A, and the second through β about B. It will be convenient to suppose both angles measured in the same clockwise direction, which is always possible for given angular displacements. Form the triangle ABC with the angles $\frac{1}{2}\alpha$ at A and $\frac{1}{2}\beta$ at B. Let A' be the image of A in

BC, B' the image of B in CA, and C' the image of C in AB. Then the triangles $A'BC$, $AB'C$ and ABC' are all equal in all respects to the triangle ABC and to one another; they are also of opposite aspect to ABC, but of like aspect among themselves and therefore superposable.

It will now be clear that a rotation about A through the angle $2A=\alpha$ will bring the triangle $AB'C$ into the position ABC', while a second rotation about B through the angle $2B=\beta$ will bring the triangle ABC' into the position $A'BC$. But the same result of

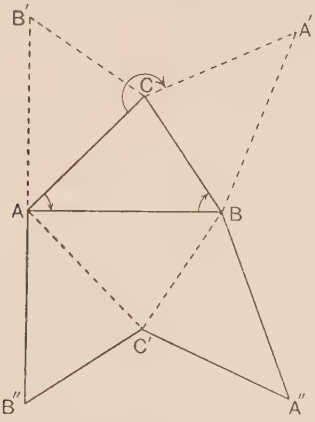

FIG. 10.

bringing $AB'C$ into the position $A'BC$ could equally be effected by a direct rotation about C through the angle

$$ACA'=360°-2C=2(A+B)=\alpha+\beta.$$

The resultant of the successive rotations about A and B is therefore a single rotation through the sum of the angles about a point C, of which the construction has been given.

Suppose now that the rotations had been made in the opposite order. Let A'' be the image of A in BC' and B'' the image of B in AC'. Then the triangles ABC, $A''BC'$, $AB''C'$ are not only equal in all respects to ABC' and to one another, but of like aspect and superposable, though of opposite aspect to ABC'. Then the triangle $A''BC'$ can be brought successively to the position ABC by a rotation about B through the angle $2B=\beta$, and then to the position $AB''C'$ by a rotation about A through the angle $2A=\alpha$; or directly by a single rotation about C' through the angle

$$A''C'A=360°-2C=2(A+B)=\alpha+\beta.$$

The effect of changing the order is therefore to change the centre of the resultant rotation from C to its image C'. It is also clear that C' will be obtained by the same rules of construction in this case as C in the former.

14. Although the result of any finite displacement in a plane can be represented by a single rotation, it is, of course, to be understood that the displacement in its intermediate stages may be any whatever, bearing no resemblance to a pure rotation. At the same time any continuous process of displacement can be conceived as a sequence of small displacements, and the effect of each can be most simply represented by a rotation through a small angle about a

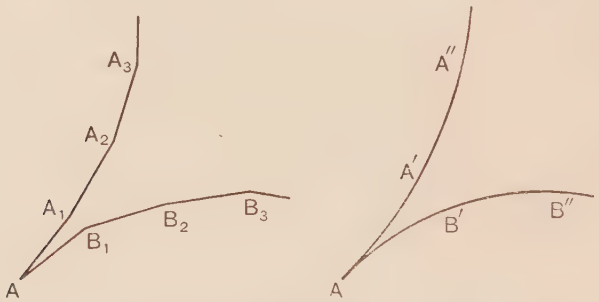

FIG. 11.

centre which is slightly shifted at each stage. For the moment these successive centres must be taken to be discrete points, forming a chain of straight links $AA_1A_2\ldots$ in space. The first small rotation about A brings a point B_1 of the body into coincidence with A_1, the second about A_1 brings a point B_2 of the body into coincidence with A_2, and so on. Thus to the chain $AA_1A_2\ldots$ fixed in space corresponds another chain $AB_1B_2\ldots$ fixed in the body, and the motion of the body is represented by this latter chain turning into coincidence link after link with the former. The process clearly requires that corresponding links in the two chains shall be equal in length; the angular deviations of each link from the preceding are small but independent of the corresponding deviation in the other chain. Hence corresponding total lengths in the two chains are equal, but the shapes of the two chains are unrelated.

For the sake of clearness the centres of successive small rotations have been treated as separate points. The result will be discontinuous motion, more closely resembling a given continuous motion the more the centres are multiplied in number and the links reduced

in length. When the continuous motion of a body in the plane is perfectly reproduced the chains become continuous curves, the one $AA'A''$ fixed in space, the other $AB'B''$ fixed in the body, the body so moving that $AB'B''$ in all positions touches the curve $AA'A''$. The curves are independent as regards shape, but the equality of corresponding lengths is preserved, so that if B' is the point of AB'' which comes to contact with A' on AA'', the arcs AB', AA' are equal. This means that the curve AB'' rolls without slipping on the curve AA'' and the motion is defined when these two curves are known.

15. When in the course of this rolling motion a point such as B' comes into coincidence with A', the point A' or B' is the point about which the body is at the instant turning. It is called the *instantaneous centre*, and is the point of the body which is at the instant stationary in space. The curve $AA'A''$ fixed in space, which may now be defined as the locus of the instantaneous centre in space, is called the *fixed* (or space) *centrode*. The curve $AB'B''$ fixed in the body, which may similarly be defined as the locus of the instantaneous centre relative to the body, is called the *moving* (or body) *centrode*. Any point of the body is rigidly attached to the moving centrode, and its path, as resulting from the rolling of this centrode on the fixed centrode, is called a *roulette*.

Since at any instant the body is rotating about the instantaneous centre, every point of the body is then moving in a circle about that centre, in the direction of the tangent and at right angles to the line joining the centre to the point. This simple type of motion is only momentary and continually changing. But two consequences follow. If the instantaneous centre C is known, any point P of the body is moving at right angles to CP, and CP is perpendicular to the tangent, *i.e.* normal to the path of P. Conversely, if the direction in which a point P is displaced be known, the perpendicular to this direction through P contains the instantaneous centre C; if a second point P' of the body gives similar information, the centre C is determined as the intersection of two straight lines.

16. *Ex.* Two given points of a body move on two fixed straight lines: to find the centrodes.

The points A, B move respectively on the lines OX, OY. Since the displacement of A is along OX, the instantaneous centre C lies on AC, at right angles to OX. Similarly C lies on BC, at right angles to OY. Since OAC, OBC are right angles, the points $OABC$ lie on a circle, of

which OC is the diameter and D, the middle point of OC, is the centre.

Let DM be the line which bisects the chord AB at right angles, and let α be the angle XOY. Then the angle

$$BDM = \tfrac{1}{2}BDA = BOA = \alpha,$$

and $AB = 2BM = 2DB \sin \alpha$. Hence $OC = 2DB = AB/\sin \alpha$, and is a constant length. Therefore C describes a circle of radius $AB/\sin \alpha$ about O, and this circle is the fixed centrode.

Since $DB = DA = \frac{1}{2}AB/\sin \alpha$, the isosceles triangle DAB is invariable and D is a definite point of the area moving with AB. But $DC = DB$

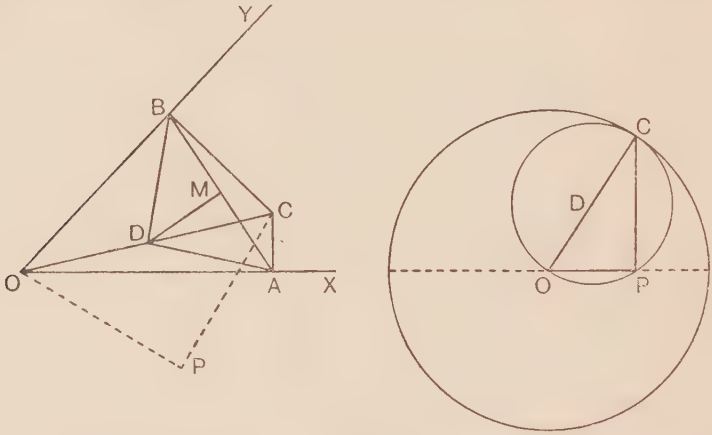

FIG. 12.

and is of constant length. Hence in the moving area C describes a circle of radius $DC = \frac{1}{2}AB/\sin \alpha$ about D, and this circle is the moving centrode.

Since D is the middle point of OC, the moving centrode always passes through O, and its diameter is equal to the radius of the fixed centrode. If P is any point carried on the moving centrode, of which OC is a diameter, OPC is a right angle. But, C being the instantaneous centre, the motion of P is at right angles to CP or along OP. Hence the roulette traced by P is a diameter of the fixed centrode. The roulette traced by a carried point not on the circumference of the moving centrode can be shown, not quite so easily, to be an ellipse.

17. *Ex.* The motion of a body is defined by a rod, of which one end A moves on a straight line and which passes through a fixed point B: to find the centrodes.

Let D be the foot of the perpendicular from B on the straight line, and let $BD = h$. Let C be the instantaneous centre. Since A moves along AD, CA is perpendicular to AD. The point B of the rod has no motion transverse to the rod and must therefore move in the direction of the rod. Hence CB is perpendicular to AB. The position of C is thus determined.

The moving centrode, being the locus of C relative to AB, will be defined by a relation connecting $AC = r$ with $CAB = \theta$. Now

$$AC \cos^2 CAB = AB \sin BAD = BD.$$

Hence the required relation is $r \cos^2\theta = h$, and it may be left as an exercise to plot the curve $r = h \sec^2\theta$ as a graph.

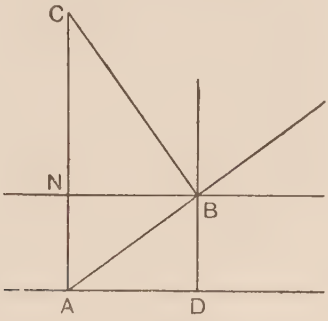

FIG. 13.

Draw the line BN parallel to AD and perpendicular to AC. The fixed centrode, being the locus of C in space, can be defined by a relation connecting $BN = x$ with $CN = y$. But, since the right-angled triangles BCN, ABN are similar, this relation is $BN^2 = NA \,.\, CN = BD \,.\, CN$,

or $$x^2 = hy,$$

which is easily graphed, and shows that the fixed centrode is a parabola.

18. The general ideas thus introduced by representing any plane motion as the result of the rolling of one specified curve on another may now be applied to purely geometrical conceptions which will afterwards be useful. Notice that, as the rolling of one centrode on another gives rise to a certain type of motion, so conversely the centrode moving in a continuous specified way may be considered to generate the fixed centrode by its contacts. A curve which is thus generated is called the *envelope* of the moving curve, which may, in a sense more general than the centrode, gradually change its form as well as its position.

Let PP' be an ordinary curve, PT, PN the tangent and normal at P, $P'T'$, $P'N'$ the tangent and normal at P'. These pairs of lines, being at right angles, may be considered a frame to which a moving body is attached. The curve and tangent are not the fixed and moving centrodes, because the tangent is not rolling on the curve but slipping at the same time, the point of contact (considered as a fixed point of the body) moving along the curve. This motion along the curve in the direction of the tangent shows in fact that

the instantaneous centre lies on the normal, and, as this line is fixed in the body and contains the instantaneous centre, the normal is the moving centrode. The type of motion of the body in this case is less general than that before considered from the circumstance that the moving centrode is a straight line. But it equally follows that a fixed centrode exists, which is the envelope of the moving normal. This curve, to which the normal to the curve PP' is a tangent, is called the *evolute* of PP'. Reciprocally, the curve PP' is called an *involute* in relation to its evolute ; it will be seen immediately that, while a curve possesses one evolute, it possesses an unlimited number of involutes.

By the common property of centrodes, the normal PN rolls without slipping on the evolute. The point P therefore moves on

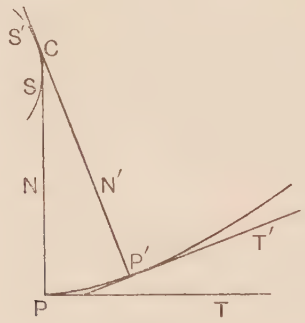

FIG. 14.

its curve PP' like the end of a taut string which is being unwound from the evolute. But while in this process the point P is a fixed point in the string, the original choice of a point on the unwound, straight portion of the string is quite arbitrary, and by varying the choice of this point any number of involutes can be drawn possessing the same evolute in common. Another consequence follows from this generation of the involute by unwinding a string. Let S, S' be the points of contact of the normals at P, P' with the evolute. The length of the arc SS' of the evolute is clearly the difference between the lengths of the normals $P'S'$ and PS. It is also evident that different involutes have not only common normals, but intercept equal lengths on each normal, so that they are parallel curves.

19. These simple deductions from the consideration of plane motion lead to the property of curvature in the geometry of the

curve. In fact, the point of contact S of the normal at P with the evolute is called the *centre of curvature*, and the length SP on the normal is called the *radius of curvature* at P. Thus the centre of curvature is the point about which the normal at P is instantaneously turning, and may be determined in practice by finding the intersection of the normal at P with a second, adjacent normal and making the second move towards coincidence with the first, thereby causing the three points C, S and S' of the figure to coalesce in the required centre. From this point of view the evolute of a curve may now be defined as the locus of its centres of curvature.

The circle drawn with centre S (the centre of curvature) and passing through P is called the circle of curvature at P. Its radius is the radius of curvature, and it touches at P the same tangent PT as the curve. Its local relation to the curve is important.

In the figure two adjacent normals at P, P' intersect in C and touch the evolute at S, S'. It has been seen that $P'S' = PS + \text{arc } SS'$, by the property of centrodes, and hence

$$CP - CP' = CS + CS' - \text{arc } SS'.$$

The angle $PCP' = \psi$ between the normals is small, and by analogy with a circular arc in place of SS', or even by substituting the chord for the arc, it is easily seen that the difference between the arc and the sum of the tangents from C is of the order of $SS' \,.\, \psi^2$ or ψ^3, whereas if any other centre C' were taken on CP, the difference between $C'P$ and $C'P'$ would be of the order ψ^2. Moreover, in general, no circle can be drawn through P, P' so as to touch both PT and $P'T'$, for this would require it to satisfy four conditions instead of three. As things are, a circle drawn with centre C and radius intermediate between CP and CP' will cross the curve between P and P', and its normal distance from the curve at P or P' will be of the third order in ψ. This circle in the limit, when P' approaches P and the points C, S', S coalesce, clearly becomes the circle of curvature at P. It retains in general its property of crossing as well as touching the curve at P, and over a short arc in the neighbourhood of P it approximates to the curve more closely than any other circle, its departure from the curve at any point being of the third order when the arc to the point is a small quantity of the first order.

20. This insistence on the order of small quantities will be better understood by considering a method, due to Newton, of finding the

radius of curvature. Let C be the centre of the circle of curvature, its radius being $CP=\rho$. Let QQ' be a chord parallel to CP, cutting the tangent PT at right angles in N, and let $PN=x$, $Q'N=y$. Then $PN^2=NQ' \,.\, NQ$, and, since $NQ=2\rho-NQ'$,

$$x^2=y(2\rho-y),$$

or

$$2\rho=\frac{x^2}{y}+y.$$

If therefore $PN=x$ is a small quantity in relation to ρ, as it may be chosen, y must be a small quantity of the same order as x^2 or small in relation to x. This result may be written

$$\rho=\text{Lim.}\,\frac{x^2}{2y}, \quad x\to 0,$$

meaning the ultimate value of this fraction as x is indefinitely diminished. It has been obtained by reference to the circle of

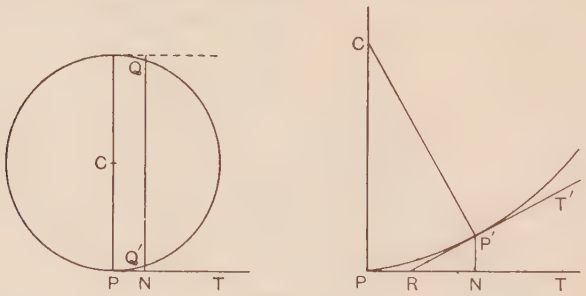

FIG. 15.

curvature only. But if P' is the point of the curve corresponding to Q', the difference between $P'N$ and $Q'N$ is of the third order when the arc PQ' or PN is a small quantity of the first order. Hence the ordinate $P'N$ of the curve can be substituted for the ordinate $Q'N$ of the circle or for y in the given limit, and therefore the radius of curvature at P can be found by the formula, when the relation between x and y is known for the curve.

Ex. To find the radius of curvature at the origin when $y^2-x^2=12x$. In this case, on division by $2x$,

$$\frac{y^2}{2x}-\frac{1}{2}x=6.$$

The axes are reversed, the axis of y being the tangent to the curve. But equally the limit of $y^2/2x$ is the radius of curvature. Hence, as $x\to 0$, $\rho=6$. Newton's method can be directly applied when the term of the lowest degree in the equation is either x or y.

21. Another method of finding the radius of curvature may be indicated. Let the tangents PT, $P'T'$ meet at R at the angle ψ, which may be regarded as the angle through which the curve has turned between P and P'. The normals CP, CP', being perpendicular to the tangents, meet at the same angle ψ. From its closeness to the circle of curvature, the arc of the curve $PP' = s = \rho \cdot \psi$ approximately. As P' recedes from P, the approximation becomes less close, and to be accurate it is necessary to diminish the arc indefinitely and to write

$$\rho = \text{Lim.} \frac{s}{\psi}, \quad s \to 0.$$

This is in line with the natural meaning of *curvature* as the rate at which the curve turns as the arc increases. This rate is measured by the ratio ψ/s when both ψ and s are indefinitely diminished. On this view curvature is estimated by the *reciprocal* of the radius of curvature, $1/\rho$, an appropriate measure, for when the curve is flat and but slightly curved the circle of curvature is large, but when the arc is sharply curved the circle is small.

22. The simplest curves which can be chosen as centrodes are naturally straight lines and circles, but the resulting roulettes traced by carried points are not in general specially simple curves, though they may be considered to belong to one general family. There is, however, one particular curve of special interest, which deserves notice. This is the *cycloid*, traced by a point on the circumference of a circle which rolls on a straight line. Here the straight line is the fixed centrode, and the circle is the moving centrode.

Let AA' be the fixed straight line, on which the circle, centre G, is rolling. Let FGE be the diameter through the point of contact E. The tracing point P was itself the point of contact at A, so that the arc $PE = AE = a\theta$, where a is the radius of the circle and θ is the angle PGE. It is evident that every fixed point on the circle will trace an exactly similar curve (cycloid), the only difference between the curves being a horizontal displacement in the direction of AA'. In particular, the cycloid traced by P' at the opposite end of the diameter through P is the same as the cycloid traced by P and shifted through the distance πa equal to half the circumference of the circle, while each cycloid returns to the fixed line AA' at intervals equal to the whole circumference. The curves AB, $A'B'$ show one half of each of these cycloids, and can be completed by symmetry.

Here $AA'=BB'=\pi a$, and $AB'=A'B=2a$. The highest point B is the *vertex* of the cycloid AB.

Draw $P'C$ parallel to the diameter FGE to cut PE in C. Then $P'C/GE=PC/PE=PP'/PG=2$, or $PC=2PE$ and $P'C=2GE=FE$. Hence the line EC is obtained by displacing FP' parallel to itself through a vertical distance equal to the diameter of the circle. It follows that the locus of C is the locus of P', and the envelope of PEC is the envelope of FP', similarly displaced.

But in the rolling motion the point of contact E is the instantaneous centre. Hence PE is the normal at P to the cycloid AB, and, since EPF in a semicircle is a right angle, PF is the tangent at P. Similarly $P'E$ is the normal and $P'F$ is the tangent at P' to the cycloid $A'B'$. In other words, the cycloid $A'B'$ is both the

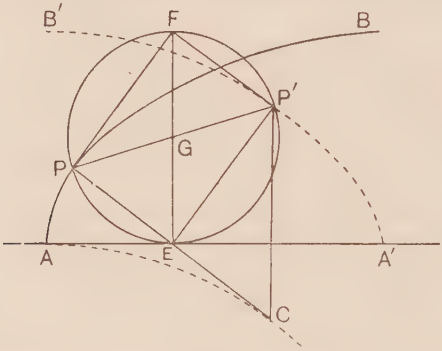

FIG. 16.

locus of P' and the envelope of FP'. Therefore the locus of C and the envelope of PC is this same cycloid displaced vertically through a diameter. Hence the cycloid AC, being the envelope of the normal PC, is the evolute of the cycloid AB, and C is the centre of curvature of AB at P, the radius of curvature being

$$PC=2PE=2FE\sin PFE=4a\sin\tfrac{1}{2}\theta.$$

In relation to the cycloid AC, the cycloid AB is an involute, and the arc PA can be generated by winding PC like a taut string on AC. Thus the length of the arc $AC=PC=4a\sin\frac{1}{2}\theta$, where $\frac{1}{2}\theta=PFE=A'EC$, the angle which the tangent EC makes with AA'. In a half-period of rotation, the angle $PGE=\theta$ goes from 0 to 180°, so that the half-length of the cycloid, AB or $A'B'$, is $4a$. At P the tangent PF to the branch AB makes the angle $90°-\frac{1}{2}\theta$ with AA' or BB', and therefore the arc $PB=4a\cos\frac{1}{2}\theta$, while the arc

$$AP=4a(1-\cos\tfrac{1}{2}\theta).$$

23. It may not be superfluous to emphasise the distinction to be drawn between the ideas represented by the instantaneous centre and the centre of curvature. When a plane area, with any number of points conceived as rigidly attached to it, is moving in its own plane, it is at each instant turning about a certain point O, the instantaneous centre. It is necessary not to understand more by this statement than is intended, namely (1) the point O is *at the instant* motionless, and (2) every point P attached to the moving plane is moving at right angles to the radius OP. The motion of every point P is thus defined as to immediate direction, but nothing is implied as to the initial curvature of its path. In fact, if an individual point P alone is considered, any point O' on OP would serve this purpose as well as the instantaneous centre O itself, for P is equally moving at right angles to $O'P$. But O' is not stationary at the instant, and does not possess the same property as O with respect to points not lying on OP.

Not only does the point O not possess any special virtue among the points on OP with respect to the motion of an individual point P, but there is a point, the centre of curvature C, which does. For a circular motion of P about C represents not only the initial direction of the motion of P, but also the initial curvature of the path of P. Thus C lies on OP, the normal to the path of P, but does not in general coincide with O. In general, when P is a carried point, the tangent PT to the path of P is not a line also fixed in the moving body and carried with it. Only in the special case when this tangent is fixed in the body, and so defines its motion, does the centre of curvature coincide with the instantaneous centre for the motion thus defined.

24.* It is now possible to find the centre of curvature of the path of any point carried by one given curve which is rolling on another.

Let O be the point of contact of the two curves at a given instant, A the centre of curvature of the fixed curve and M the centre of curvature of the rolling curve, corresponding to O. The line AOM is a straight line which is normal to both curves at O.

The given curve is now slightly displaced by rolling. Let O' be the new point of contact, and M' the displaced position of M. Then AO' is the adjacent normal to AO in the fixed curve, and $M'O'$ is the adjacent normal to MO, carried into its new position. Hence $AO'M'$ is a straight line which is normal to both curves at O'.

* The rest of this chapter may be omitted on first reading.

To a close approximation $AO=AO'$, $MO=M'O'$ and $AM=AM'$. The lines AOM, $AO'M'$ are normals not only to the small arc OO' of the fixed curve, but also to the small arc MM' of the path of M. Thus A is the centre of curvature of the path of M at M.

But the rolling motion is not a motion of rotation about A. The instantaneous centre is the point of contact O, and the angular displacement is measured by the angle MOM' (or $MO'M'$). If P,

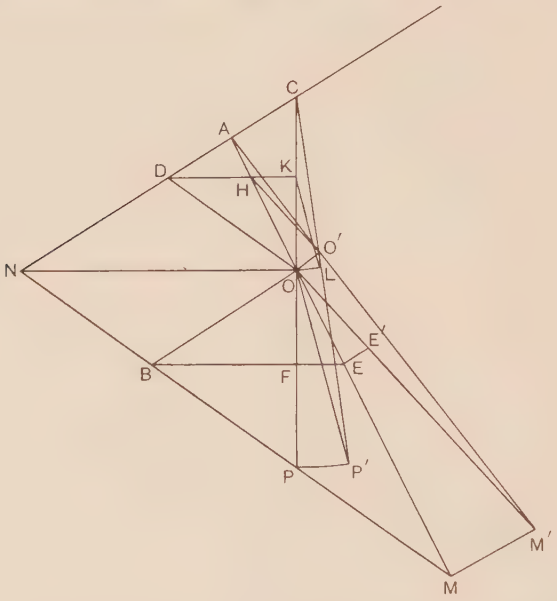

FIG. 17.

a point connected with the rolling curve, is at the same time carried to the position P', OP is normal to the curve PP', and the angle POP' is equal to the angle MOM'. Thus the centre of curvature of the path of P is some point C on OP, and CP, CP' are adjacent normals to this path.

25. The arc OO' is ultimately indistinguishable from the perpendicular from O on AM'. Let OL be the perpendicular from O on CP'. Since the angles POP', MOM' are equal,

$$OP'L+OCL=OAO'+OM'O',$$

and therefore

$$\frac{OL}{OP'}+\frac{OL}{OC}=\frac{OO'}{OA}+\frac{OO'}{OM'},$$

or

$$\left(\frac{1}{OP'}+\frac{1}{OC}\right)\frac{OL}{OO'}=\frac{1}{OA}+\frac{1}{OM'}.$$

But OL, OO' being perpendicular to CP', AM', the angle

$$LOO' = P'O'M'$$

and is ultimately equal to $POM = AOC = \theta$. Hence, when the displacement is indefinitely small,

$$\left(\frac{1}{OP} + \frac{1}{OC}\right) \cos LOO' = \frac{1}{OA} + \frac{1}{OM},$$

or, if $OA = R$, $MO = \rho$, $OP = r$, $PC = \sigma$,

$$\left(\frac{1}{r} + \frac{1}{\sigma - r}\right) \cos\theta = \frac{1}{R} + \frac{1}{\rho}.$$

This is the equation which determines σ, the radius of curvature to the path of the carried point P. It is clearly satisfied by $\theta = 0$, $r = \rho$, $\sigma = R + \rho$, which confirms the remark that C coincides with A when P coincides with M.

26. The centre of curvature C can be found by a simple geometrical construction. Let AC, MP meet in N and join NO. Let OD parallel to MN meet CN in D, and draw DHK parallel to NO to meet OA, OC in H and K. Join HO', KL. Then, OO' being parallel to MM' and OL to PP',

$$\frac{CK}{CO} = \frac{CD}{CN} = \frac{CO}{CP} = \frac{CL}{CP'},$$

whence KL is parallel to OP'. Similarly

$$\frac{AH}{AO} = \frac{AD}{AN} = \frac{AO}{AM} = \frac{AO'}{AM'},$$

whence HO' is parallel to OM'. Therefore the angle

$$OKL = POP' = MOM' = OHO',$$

and the right-angled triangles KOL, HOO' are similar. Hence $OK/OH = OL/OO' = \cos LOO' = \cos KOH$, or OKH is a right angle. But KH is parallel to ON, and thus ON is perpendicular to OK. Hence the following construction for the point C:

Join MP, and let this line cut the perpendicular ON to OP in N. Join NA. This line will cut OP in the required point C which is the centre of curvature of the path of P.

Let OB parallel to AN meet MN in B. Draw BFE parallel to NO to meet OM, OP in E and F. Then

$$\frac{OH}{OA} = \frac{ND}{NA} = \frac{MO}{MA},$$

and similarly

$$\frac{OE}{OM} = \frac{NB}{NM} = \frac{AO}{AM}.$$

Hence $$OE=OH=AO\,.\,OM/AM,$$

or $$\frac{1}{OE}=\frac{1}{OH}=\frac{1}{AO}+\frac{1}{OM}=\frac{1}{R}+\frac{1}{\rho}.$$

It has been proved incidentally that $AH\,.\,AM=AO^2$. Equally $ME\,.\,MA=MO^2$, for $ME/MO=MB/MN=MO/MA$. It has also been proved that $CK\,.\,CP=CO^2$, and with equal ease it can be seen that $PF\,.\,PC=PO^2$. A deduction from these relations is that the points F and C lie on the same side of P in all circumstances.

27. It will now be evident that the relation between any carried point P and the corresponding centre of curvature C of its path depends only on the position of the point of contact O and on R and ρ only in the combination $R^{-1}+\rho^{-1}$, that is, on OE. In the

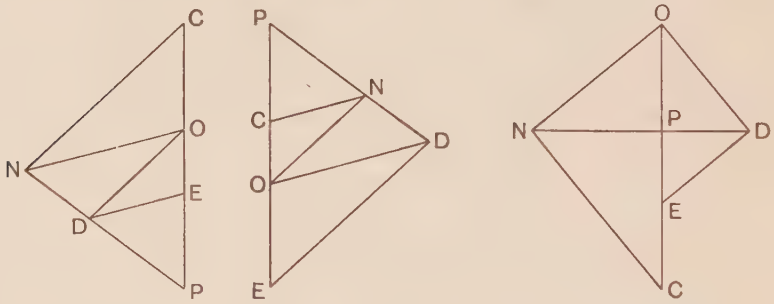

FIG. 18.

case which has been taken as standard, the fixed and rolling curves are convex to each other. If the curvature of the fixed curve is in the opposite direction, R must be taken negative. Then, if $R+\rho$ is still positive, OE is negative and must be measured in the opposite direction to OM; but, if $R+\rho$ is negative, OE is positive and to be measured in the same direction as OM.

When the points O and E are given in position, and the tracing point P and the corresponding centre of curvature C lie on OE, the general relation $PF\,.\,PC=PO^2$ becomes $PE\,.\,PC=PO^2$. This is a simple way of finding C when P is given. In this case $\theta=0$ and

$$\frac{1}{OP}+\frac{1}{OC}=\frac{1}{OA}+\frac{1}{OM}=\frac{1}{OE}.$$

Hence any corresponding pair of points P and C on OE may be substituted for M and A for the purpose of finding E, or for making the geometrical constructions described above. The construction of such a pair when OE is given may be made by the following rule :

Join any point D to O, E and P. Let ON parallel to DE meet PD in N. Then NC drawn parallel to DO meets PO in C, the centre of curvature corresponding to P.

In any one of the three distinct figures which result from this rule according to the position of P, it is evident that

$$PE/PO = PD/PN = PO/PC,$$

or $PE \,.\, PC = PO^2$, which gives the required relation between P and C.

Alternatively, in order to obtain the point E when a pair of corresponding points P, C are given, it is only necessary to interpret the same figure differently to have this rule:

Join any point N to O, P and C. Let OD parallel to NC meet NP in D. Then DE drawn parallel to NO meets PC in E.

28. On any other line through O, the relation $PF \,.\, PC = PO^2$ is secured by a similar construction when F takes the place of E, the point F being the projection of E on the line OP.

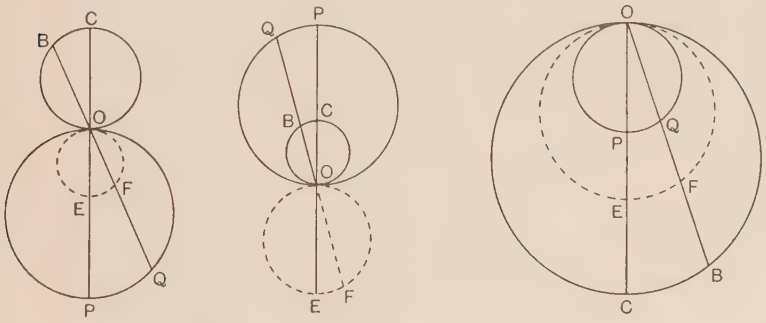

FIG. 19.

Let C be the centre of curvature corresponding to a point P on OE, and let B be the centre of curvature corresponding to a point Q on OF, which makes the angle θ with OE. Then

$$\left(\frac{1}{OQ}+\frac{1}{OB}\right)\cos\theta = \frac{1}{R}+\frac{1}{\rho} = \frac{1}{OP}+\frac{1}{OC}.$$

If therefore $OQ = OP \,.\, \cos\theta$, $OB = OC \,.\, \cos\theta$. Now these relations are secured by drawing circles on OP and OC as diameters, touching at O. For, if QOB is drawn such that $POQ = \theta$ to cut the circles in Q and B, the angles OQP, OBC in semi-circles are right angles, so that $OQ = OP \,.\, \cos\theta$ and $OB = OC \,.\, \cos\theta$. Hence the point B is the centre of curvature corresponding to Q.

Thus for all carried points which lie on a circle passing through the point of contact O of the fixed and rolling curves, which has its centre on the line OE, the centres of curvature lie on another circle also passing through O, of which the centre also lies on OE. The relation between these circles and the circle on OE as diameter is shown in three distinct figures corresponding to the preceding set.

29. These figures show how the position of the centre of curvature depends on whether the point P or Q is inside or outside the circle on OE as diameter. This circle will be called the critical circle, and it remains to see what is the result when the carried point lies on it. Since $PC \,.\, PE = PO^2$, it is evident that PC increases without limit when PE is diminished without limit, or PC becomes infinite when P coincides with E. This also follows, when $OP = OE$, because $1/OP + 1/OC = 1/OE$. A reference to fig. 17 (p. 22) shows the same thing in another way. When OM is displaced to OM', E is displaced to E' on OM', which is parallel to HO'. Since $HO = OE$, EE' is equal and parallel to OO'. Hence $O'E'$ is parallel to OE, and these lines meet in the centre of curvature corresponding to E, so that this point is at infinity. Thus, when the carried point lies on the critical circle, the locus of the corresponding centre of curvature is an infinite circle, which is the line at infinity.

30. The radius of curvature of a roulette being infinite at a point where it meets the critical circle, the curvature at the point is

FIG. 20.

zero. Now this means in general that the curvature changes sign through the value 0. Since the curvature is measured by the limit of ψ/s, or the rate of turning of the tangent as the point of contact moves along the curve, it is to be inferred that the tangent begins to turn in the opposite direction at the point now considered. Such a point is called a point of inflexion and is of the nature here represented. The direction in which the curve is concave, or convex, is reversed at the point. Being the locus of these points of inflexion on the roulettes, the critical circle is sometimes called the circle of inflexions.

While a point on the roulette traced by a point on the critical circle is generally of this kind, there are exceptions. When, for instance, the two centrodes are circles, it is evident that the roulette traced by the point E is symmetrical with respect to the line OE, and a point of inflexion is precluded. The simple fact is that at a point of the roulette where the curvature vanishes the curve is flat. But there are degrees of flatness. The lowest and simplest degree is represented by a point of inflexion. Higher degrees of flatness may be associated with points apparently resembling a point of inflexion or with points not apparent to the eye, where the concavity of the curve is in one definite direction. At such points the question of this direction can only be decided by taking into consideration a higher degree of approximation than has been necessary above. For applications to statics, in relation to which the critical circle has been called the circle of stability, the results already found are sufficient for all points which do not lie on this circle. But when points on the critical circle are in question, a closer examination of the conditions, represented by the forms of the centrodes near their point of contact, will be required.

31. The general motion of a body in three dimensions, with its six degrees of freedom, is naturally more complicated than in two dimensions. If, however, one point of the body is fixed, three degrees of freedom are removed and the conditions are greatly simplified.

Let O be the fixed point. If a sphere is drawn about O as centre, with any given radius, it will cut the body in points which will always remain on the sphere, and in a fixed relation to one another, as the body moves. The motion of the body is fully represented by the motion of a given area on this sphere, and presents a two-dimensional problem, which differs from that already considered only in the substitution of a spherical surface for a plane. The analogy is complete when straight lines are replaced by great circles, or the traces on the spherical surface of planes which pass through the centre of the sphere.

In the first place, any displacement can be effected by a single rotation about an axis through O.

Let the points AB be displaced on the surface of the sphere to $A'B'$, so that the arcs AB, $A'B'$ are equal. Join AA' by the trace of the plane AOA', and let α be the angle $A'AB$, or the angle between the planes AOA', AOB. Similarly, let β be the angle $AA'B'$, both

these angles being measured from AA'. Draw the great circle arcs AC, $A'C'$, making equal angles $\frac{1}{2}(\alpha-\beta)$ in the same direction with AB, $A'B'$, and of equal length. Thus the spherical triangles ABC,

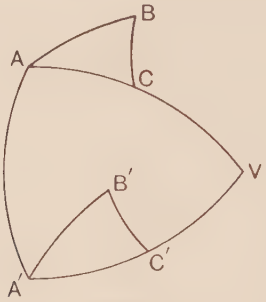

FIG. 21.

$A'B'C'$, having equal pairs of sides containing equal angles, will be brought into coincidence when AC coincides with $A'C'$. But the angle $A'AC=\alpha-\frac{1}{2}(\alpha-\beta)=\frac{1}{2}(\alpha+\beta)$, and the angle

$$AA'C'=\beta+\tfrac{1}{2}(\alpha-\beta)=\tfrac{1}{2}(\alpha+\beta).$$

Hence $A'AC=AA'C'$, and if AC, $A'C'$ meet in V, it is clear that $VA=VA'$. Therefore a rotation of the surface of the sphere about V, or about the axis OV through the fixed point of the body, will bring AC into coincidence with $A'C'$, B with B', and similarly every point of the body into its displaced position.

32. The effect of two successive rotations about the axes OA, OB given in space must be a single rotation about an axis OV, and this

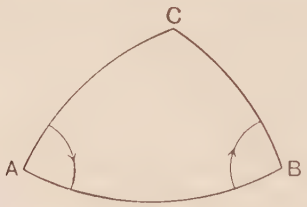

FIG. 22.

can be found without difficulty. On the surface of the sphere about the centre O it is possible to construct a figure exactly analogous to fig. 10 (p. 11), in which the triangles are now spherical triangles, with sides which are the traces of planes through O. The sides are measured by the angles which they subtend at O, and the angles of the triangles represent the angles between the planes of which the adjacent sides are the traces.

A rotation through the angle $\alpha=2A$ about the axis OA is followed by one through $\beta=2B$ about OB. The angles A and B being thus given, the triangle ABC can be constructed. The steps of the argument in the plane analogue can be easily followed to the point which shows that OC is the axis of the equivalent single rotation, and that the angle of rotation is $360°-2C$. This is not, however, equal to $2(A+B)$, because the three angles of a spherical triangle are together greater than two right angles. In fact, by spherical trigonometry, if c is the side AB,

$$\cos C=\sin A\sin B\cos c-\cos A\cos B,$$

or, if $\gamma=360°-2C$ is the resultant rotation about OC,

$$\cos\tfrac{1}{2}\gamma=\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta-\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cos c.$$

It will be evident that the combination of finite rotations in no way resembles the combination of vectors.

33. It is the characteristic property of the sphere that an area of its surface can be moved on the surface without change of shape, just as a plane area can be moved in its plane. The circumstances are entirely similar. In both cases a finite displacement can be reproduced by a simple rotation. Though the intermediate stages of an arbitrary displacement cannot in general be represented in this way, yet each stage is one of a series of elementary displacements, the effect of which is most simply represented as an elementary rotation. As in the plane analogy, the whole process can be conceived as brought about by the rolling of one spherical curve or centrode fixed in the body on another spherical curve or centrode fixed in space. The path of any point of the body is thus a roulette, but in this case a spherical roulette.

Now the point O is a point fixed in space and in the body, and if O be joined by a straight line to every point on the fixed and moving spherical centrodes, two cones (in the general sense of the word) are formed, each with its vertex at O. The centrodes being always in contact at some point C, the two cones are in contact along the generator OC. Hence the general motion of a body, of which the point O is fixed, can be represented by the rolling of one cone, with vertex O and fixed in the body, on another cone, with vertex O and fixed in space.

34. The general motion of a body, of which no point is fixed, can now be shortly considered. Any displacement can be effected

in two stages, first by a translation without rotation so as to bring any point O of the body to its final position O', and secondly by a displacement in which O' remains fixed. The latter movement can be effected by a rotation about some axis $O'V$.

Draw a plane $PO'M$ through O' perpendicular to $O'V$ and a parallel plane OT through O, and the line OP parallel to $O'V$. In the rotation about $O'V$ the points first in the plane OT, and then brought into the plane PM, remain in the plane PM. It is therefore clear that the desired result might be equally effected by a suitable displacement in the original plane OT, involving a rotation about an axis parallel to $O'V$, followed by a translation of the body parallel to $O'V$, so as to bring the plane OT after rotation into the position of the plane PM. The displacement is thus expressed as a *screw*

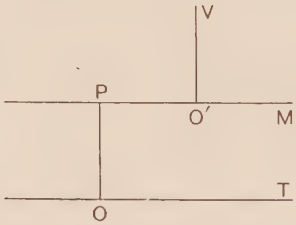

FIG. 23.

movement, or a rotation about a certain axis together with a translation along it. Of the six degrees of freedom of the body, two are represented by the rotation and the translation, while the remaining four are involved in the position of the axis, which needs four coordinates for its determination. The ratio of the translation OP to the angle through which the body is rotated is called the *pitch* of the screw displacement.

In this way the effect of any displacement of a body is very simply expressed, whatever the process of displacement may actually have been. In the particular case of a pure rotation the pitch of the screw is zero, in the case of a pure translation it is infinite. When the actual process is regarded as a succession of elementary displacements, each stage can be represented as a small screw displacement, implying a small translation along, and rotation about, some definite axis. In a continuous process therefore the screw motion takes place with varying translation and rotation about an axis which is continuously changing position both in the body and in space. At any given instant the axis has a definite position and

is the *instantaneous axis* ; the pitch also has a definite instantaneous value.

As the instantaneous axis moves in the body and in space it traces two surfaces, which are called the moving and fixed axodes. These surfaces, like the centrodes, are always in contact, but the rolling of the one on the other is accompanied by slipping in the direction of the generating axis.

EXAMPLES.

1. The following journeys are taken successively: (1) 62 miles E.N.E., (2) 37 miles N.N.W., (3) 46 miles S.W., (4) 15 miles N.W. Plot them on a convenient scale in the order given, and also from the same starting point in the order (2), (4), (1), (3). What is the length and direction of the shortest journey home ?

2. $ABCD$ is a tetrahedron. By considering the resultant of three vectors represented completely by AB, AC, AD, show that the lines joining the middle points of pairs of opposite sides of the tetrahedron meet in a point and bisect one another.

3. Vectors represented by AB, AD, CB, CD act along the sides of the quadrilateral $ABCD$. Prove that their resultant passes through the middle points of the three diagonals of the quadrilateral.

4. Equal vectors, n in number, are laid down end to end, the direction of each making the angle α with the one before. Prove that the length of the resultant vector is $\sin \frac{1}{2}n\alpha / \sin \frac{1}{2}\alpha$, if each is of unit length, and that it makes the angle $\frac{1}{2}(n-1)\alpha$ with the vector first laid down.

Hence show that the vectors form a closed polygon if $\alpha = 2r\pi/n$, where $r = 1, 2, \ldots$, or $n-1$. Draw the three distinct polygons when $n=7$.

5. The components of a vector on given rectangular axes are X, Y. Show that, in order to obtain rectangular components of the same vector in the ratio $m : n$, the axes must be turned through the angle θ or θ', where

$$\tan\theta = \frac{nY - mX}{mY + nX}, \quad \tan\theta' = \frac{mY - nX}{nY + mX}.$$

Verify that the angle between these two sets of axes after rotation is independent of X and Y, and explain the reason.

6. One point of a body in space lies on a fixed line, and a line of the body passes through a fixed point. How many degrees of freedom does the body possess ?

7. A body moves in space with two lines of the body always parallel to a fixed plane. How many degrees of freedom has it ?

8. If a rotation through α about A, followed by a rotation β about B, is equivalent to a rotation $\alpha+\beta$ about C (all in one plane), prove that the perpendicular from C on AB is

$$AB \,.\, \sin\tfrac{1}{2}\alpha \sin\tfrac{1}{2}\beta / \sin\tfrac{1}{2}(\alpha+\beta),$$

and that the foot of this perpendicular divides AB in the ratio

$$\tan\tfrac{1}{2}\beta : \tan\tfrac{1}{2}\alpha.$$

9. When the rotation about B is equal and opposite to the rotation about A ($\beta = 360^\circ - \alpha$), show how the construction of § 13 fails. Interpret the result as a translation without rotation, and show that the common displacement of every point is $2AB \,.\, \sin\frac{1}{2}\alpha$ in the direction making the angle $90^\circ - \frac{1}{2}\alpha$ with AB.

10. An angle of given size moves in a plane so that each of its sides is in contact with a given curve. Show that the circle through the vertex and the points of contact of the two sides of the angle in any position touches the locus traced by the vertex.

11. Two intersecting lines of a body moving in a plane pass through two fixed points. Show that the centrodes are circles, the radius of the moving centrode being double the radius of the fixed centrode.

12. A circle rolls on another inside it and of half its diameter. Prove that any given diameter of the rolling circle meets the circumference of the fixed circle in a fixed point.

13. An area moves in a plane, with two straight lines in it always in contact with two fixed circles. Prove that both the centrodes are circles, and that the radius of the moving centrode is double the radius of the fixed centrode.

14. An area moves in a plane with one point on a fixed straight line while a straight line of the area intersects this line in a fixed point. Prove that the moving centrode is a parabola.

15. A point P of an area in a plane moves on the fixed line a, and a line p of the area passes through a fixed point A. If the perpendicular distances of P from p and of A from a are equal, prove that the centrodes are equal parabolas.

16. An area moves in a plane so as to satisfy given conditions. Explain in what sense the moving area and the fixed plane can be interchanged. Illustrate by deducing a corresponding proposition from any one of the Examples 11-14.

17. If through each point of a curve a line of given length be drawn, making a constant angle with the normal to the curve, show that the normal to the locus of the extremity of this line passes through the corresponding centre of curvature of the given curve.

18. Find the radius of curvature of the following curves at the origin :

(i) $ax^2 + 2hxy + by^2 + 2gx = 0.$

(ii) $(x^2 + ay)^2 + (y^2 + bx)^2 = c^3 y.$

(iii) $x^3 + y^3 = 3axy.$

19. The tangent at the vertex of a cycloid being taken as the axis of x, and the inward normal at the same point as the axis of y, show that the coordinates of a point on the cycloid are

$$x = a(2\psi + \sin 2\psi), \quad y = a(1 - \cos 2\psi),$$

where a is the radius of the rolling circle, and ψ is the inclination of the tangent at (x, y) to the tangent at the vertex. Show also that

$$s = 4a \sin \psi$$

is the arc of the cycloid measured from the vertex, and that

$$\rho = 4a \cos \psi$$

is the radius of curvature.

20. If a variable circle touch a given cycloid, and also touch the tangent at the vertex, prove that the locus of its centre is a cycloid.

21. Show that the coordinates of a point on the involute of a circle of radius a are

$$x = a(\cos \psi + \psi \sin \psi), \quad y = a(\sin \psi - \psi \cos \psi),$$

when the axes are taken through the centre of the circle, ψ is the inclination of the tangent at the point (x, y) to the axis of x, and the involute leaves the circle at the point $(a, 0)$.

22. When a circle rolls on the inside of a circle of double its radius, show that the critical circle coincides with the moving centrode. How does this case illustrate the curvature of a roulette where it meets the critical circle ?

23. Two points P_1, P_2 of a moving body describe given curves, so that the corresponding centres of curvature C_1, C_2 are known. Show how to find the instantaneous centre and the critical circle.

24. A body can move with one point O fixed. OA, OB, OC are three axes fixed in space. Show that a rotation of the body in the plane AOB (that is, about an axis through O normal to this plane) through twice the angle AOB, followed by a rotation in the plane BOC through twice the angle BOC, is equivalent to a rotation of the body in the plane AOC through twice the angle AOC.

CHAPTER II

LINEAR MOTION OF A POINT

35. In the preceding chapter the idea of time was not entirely absent. A displacement implies an ordered succession of positions, each of which corresponds with an instant of time. The instants are therefore ordered, but the conception of time as a continuous quantity measurable on a uniform scale is not involved, because the comparison of different intervals of time is not required in the purely geometrical treatment. To establish relations between displacements in space and a measured time scale is the function of Kinematics.

For the present purpose the ordinary intuitive notion of time, expressible in any recognized unit, will suffice. The motion of a geometrical point will be considered in the first instance, and that in a plane only. Indeed, as a preliminary, it is necessary to deal with the motion of a point along a given fixed straight line.

36. The simplest possible type of motion is that in which the point moves along the line with a uniform velocity. This means that the displacement of the point is directly proportional to the time, or the displacement divided by the corresponding time is a constant ratio, which is the velocity of the point. If the displacement is measured by l units of length, and the corresponding interval by t units of time, then the velocity is v units of velocity, where

$$v \text{ units of velocity} = \frac{l}{t} \cdot \frac{\text{unit of length}}{\text{unit of time}} \cdot \qquad \text{.....(1)}$$

This equation subsists for all corresponding values of l and t. The units of length and time can be definitely chosen, and are then to be regarded as fundamental. The unit of velocity represents a physical quantity of a distinct kind, and may also be chosen arbitrarily and independently. But there is a clear advantage in making the choice in such a way as to establish the simplest form of relation between

the three *measures* v, l, and t. This is effected by making the derived unit of velocity such that $v=1$ when $l=1$ and $t=1$, so that

$$1 \text{ unit of velocity} = \frac{1 \text{ unit of length}}{1 \text{ unit of time}}. \quad \ldots\ldots\ldots\ldots\ldots(2)$$

In words, this unit of velocity is the velocity of a point moving uniformly one unit of length per unit of time. With this understanding the comparison of (1) and (2) leads to the simple relations

$$v=\frac{l}{t}, \quad l=vt, \quad t=\frac{v}{l} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

These equations are algebraically true when the velocity is uniform and the three units of velocity, length and time, are consistent.

For immediate purposes the unit of length will be taken to be the foot, or one-third of the British standard yard. Similarly, the unit of time will be the mean solar second, which may be defined as the second given by an ordinary clock which has a permanently uniform rate such that the clock keeps a correct account of days (of 24 hours) for an infinite time. To ensure this rate is a problem of technical astronomy; the astronomer is ultimately responsible for providing a correct measure of time.

When these units are taken as fundamental the consistent unit of velocity is one foot per second, and is written 1 ft./sec. The reduction from any other units is easily effected, as in the following example.

Ex. The velocity of a train is 60 miles per hour. This is

$$60 \frac{\text{mile}}{\text{hour}} = 60 \cdot \frac{5280 \text{ ft.}}{60 \times 60 \text{ sec.}}$$
$$= 88 \text{ ft./sec.}$$

37. In general the motion of the point along the line is not uniform, and its velocity is variable from instant to instant. This quantity may then be defined:

Def. The velocity of a point moving on a line at any instant is the rate of its displacement, or change of position, at that instant.

This definition needs explanation, because the word *rate*, which occurs in it, must be clearly understood.

In order to make the displacement of a point P measurable along a line, a fixed point O must be marked on the line. The position of P at any time t is defined by the length $OP=x$. When the motion of P is defined, there must be a certain way of expressing

x in terms of t, so that when t is given any particular value, x also has a certain corresponding value. This must be a single value, because P cannot occupy two positions at the same time. Further, if OP changes from x to x' in the interval from t to t', P must pass through all intermediate positions in the course of the interval. In mathematical language, x is a single-valued, continuous function of t, and the relation may be expressed in the form $x=f(t)$, where $f(t)$ stands for a definite mathematical expression involving t.

When the motion of P is thus defined, $x=f(t)$ at the instant t, and at the subsequent instant t', $x'=f(t')$, the result of substituting t' for t in the expression $f(t)$. The interval $t'-t$ may be taken so short that the corresponding displacement $x'-x$ is entirely in one direction. The average velocity over the interval is therefore

$$\frac{x'-x}{t'-t}=\frac{f(t')-f(t)}{t'-t},$$

for a uniform velocity of this amount would displace P through the same distance in the same time. When $f(t)$ is a simple expression, this average velocity is easily calculated.

Ex. The motion is defined by the relation $x=at^3$, where a is constant. Here $f(t)=at^3$, $f(t')=at'^3$, and the average velocity over the interval $t'-t$ is

$$\frac{x'-x}{t'-t}=\frac{at'^3-at^3}{t'-t}$$
$$=a(t'^2+tt'+t^2).$$

38. The velocity being variable, the actual velocity during the interval is partly greater and partly less than the average velocity, and its deviations from the average will naturally be the greater the longer the interval considered. To obtain the true velocity at the time t the interval must be diminished without limit. The velocity has been expressed by a ratio which appears to approach the meaningless form 0/0. But if the division by $(t'-t)$ is first performed, it is easy to find the result of making t' approach t without limit. Thus, in the case of the above example, this result may be written :

$$a(t'^2+tt'+t^2)\to 3at^2, \quad \text{as } t'\to t,$$

and $3at^2$ is the velocity of the point at the time t, when $x=at^3$.

To derive the velocity of a point, when the displacement x is expressed in terms of the time, is thus seen to involve a limiting

process. Within the restricted range of problems to be treated here this process can be effected with great ease from first principles. But it will be evident that the similar calculation of rates of change, without any restriction in the form of the expressions, is a matter of great practical importance. To provide systematic methods for this purpose is the first object of the Differential Calculus. Although the details of this Calculus will not be required it will be convenient to introduce its notation.

39. The velocity of P at the time t, being the rate of change of $OP=x$, may be denoted by $\dot{x}$, according to Newton's fluxional notation. The interval of time, $t'-t$, will be denoted by δt, while the corresponding displacement, $x'-x$, will be denoted by δx. These quantities, so expressed, are finite, but generally to be regarded as small. The definition of velocity is now expressed by

$$\dot{x}=\text{Lim.}\,\frac{\delta x}{\delta t}\,(\delta t\to 0)=\frac{dx}{dt},$$

where the third form is not to be regarded as a mere fraction, but as a purely conventional way of writing the limit which precedes it. It is the ordinary symbol for what is called the derivative of x with respect to t, and may readily be found in very simple cases by the method outlined and illustrated above. Otherwise a slight variation can be made by putting $t'-t=\tau$, or $t'=t+\tau$. Then the velocity, or derivative of $x=f(t)$, can also be expressed in the equivalent form,

$$\dot{x}=\underset{\tau\to 0}{\text{Lim.}}\,\frac{f(t+\tau)-f(t)}{\tau}=\frac{dx}{dt}=f'(t),$$

which conforms with the more usual definition of the derivative, $f'(t)$, of $f(t)$. Clearly there is no need to doubt the existence of the derivative in cases where the existence of a velocity is physically obvious. This consideration should allay difficulties which belong more properly to abstract mathematics. Both x and t are here treated as variable quantities, and when related, as in this case, x is called the *dependent* variable, t the *independent* variable. The part played by t as the independent variable in mechanics generally is intimately related with the ideal conception of time.

40. It is instructive to consider these ideas from a graphical point of view. The function $x=f(t)$ can be represented in the usual way by a graph, with t as abscissa and x as ordinate. Since x

varies continuously with t, the graph is a continuous curve. Let Q be the point (t, x) on this curve, Q' the point $(t+\delta t, x+\delta x)$, so that in the figure $QM=\delta t$, $Q'M=\delta x$. The average velocity in the interval is $\delta x/\delta t = Q'M/QM = \tan Q'QM$, where the angle $Q'QM$ is the inclination of the chord QQ' to QM or the axis OT. But if δt and δx are diminished without limit by making Q' approach

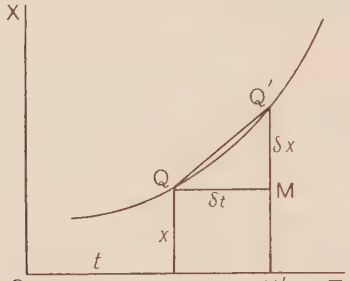

FIG. 24.

coincidence with Q, the chord QQ' will tend to coincide with the definite line which is the tangent to the graph at Q. Hence the velocity of the point P is

$$\dot{x} = \frac{dx}{dt} = \text{Lim.}\, \frac{\delta x}{\delta t} = \tan \psi,$$

where ψ is the inclination to the axis OT of the tangent to the graph at the point Q, whose coordinates represent the time and displacement at P.

It may appear that the units of time and length, being distinct in kind, can be represented by different scales on the two axes. But if these scales are relatively altered, the graph is distorted and the slope ψ of the curve changed. To interpret properly the graph when drawn, it is necessary to remember that in geometry there is only one scale common to the two axes. The unit of that common scale represents a certain unit of time on the one axis, and a certain unit of length on the other. To these units of length and time corresponds a definite unit of velocity, consistent with them in the sense already explained. The tangent of the slope of the curve is the measure of the velocity expressed in that unit. The changes alluded to are only the changes in the measure appropriate to changes in the fundamental units.

Ex. If $x=a+bt$, $\dot{x}=\tan\psi=b=\text{const.}$ The velocity is constant and the graph is a straight line.

Ex. If $x^2+t^2=a^2$, or $x=\sqrt{(a^2-t^2)}$, the graph is a circle. Here

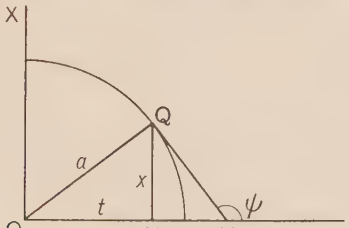

FIG. 25.

$$\dot{x}=\tan\psi=\tan(90^\circ+QOT)$$

$$=-\cot QOT=-\frac{t}{x}$$

$$=-\frac{t}{\sqrt{(a^2-t^2)}}.$$

41. *Def.* The acceleration of a point moving along a line at any instant is the rate of increase of its velocity at that instant.

Acceleration, like velocity, is a physical quantity which must be expressed in its own appropriate unit. Thus if the velocity is increased uniformly by v units in t units of time, and the result is a uniform acceleration of f units,

$$f \text{ units of acceleration}=\frac{v}{t}\cdot\frac{\text{unit of velocity}}{\text{unit of time}}. \quad \ldots\ldots\ldots\ldots(4)$$

In this dimensional form the three units may be chosen independently. By choosing them in such a way that $f=1$ when $v=t=1$, or

$$1 \text{ unit of acceleration}=\frac{1 \text{ unit of velocity}}{1 \text{ unit of time}}, \quad \ldots\ldots\ldots\ldots(5)$$

the relation between the numerical measures f, v, t assumes the simple algebraic forms,

$$f=\frac{v}{t}, \quad v=ft, \quad t=\frac{v}{f}. \quad \ldots\ldots\ldots\ldots(6)$$

The unit of velocity, like the unit of time, may be chosen arbitrarily, and the unit of length is not directly involved. But since acceleration, velocity, time and displacement are in general connected in the same problem, it is commonly advisable to keep the units of length and time fundamental and to make the unit of velocity depend on them in the way already explained. The result of uniting (2) and (5) may be expressed in the form

$$1 \text{ unit of acceleration}=\frac{1 \text{ unit of length}}{(1 \text{ unit of time})^2}.$$

This may be regarded as a convenient way of stating (if doubt is felt as to the meaning of a ratio of different kinds of physical quantity) that the unit of acceleration varies directly as the unit of length and inversely as the square of the unit of time, while the three units are associated in a particular case.

Thus if the units of length and time are the foot and second, the consistent unit of acceleration can be conveniently denoted by ft./sec.^{2}. When the velocity of a point increases uniformly at the rate of 1 ft./sec. every second the acceleration is 1 ft./sec.^{2}.

42. According to definition, acceleration bears to velocity exactly the same relation as velocity bears to displacement. If the acceleration of a point moving along a straight line be denoted by $\ddot{x}$ when the velocity is $\dot{x}$ and the displacement x, then $\ddot{x}$ can be derived from $\dot{x}$ by the same process as $\dot{x}$ from x. The treatment of variable acceleration would require no further explanation on these lines were it not desirable to provide notation for the double step from x to $\ddot{x}$.

The displacement x is expressed in terms of t by the form $f(t)$. Similarly, the velocity $\dot{x}$ is expressed by $f'(t)$, the derivative of t. At the subsequent time t' the velocity is $\dot{x}'=f'(t')$, and the average acceleration in the interval $t'-t$ is

$$\frac{\dot{x}'-\dot{x}}{t'-t}=\frac{f'(t')-f'(t)}{t'-t}.$$

A uniform acceleration of this amount, maintained through the same interval of time, would by (6) increase the velocity from $\dot{x}$ to $\dot{x}'$. The acceleration being variable, its value at the time t must be found by diminishing the interval $t'-t$ indefinitely, so that

$$\ddot{x}=\text{Lim.}\frac{\dot{x}'-\dot{x}}{t'-t}=\text{Lim.}\frac{f'(t')-f'(t)}{t'-t}, \quad (t'\to t)$$
$$=f''(t).$$

The function $f''(t)$, which results from this process, is called the second derivative of $f(t)$, $f'(t)$ being the first derivative. The process is easily performed in simple cases.

Ex. When $x=f(t)=at^3$, $\dot{x}=f'(t)=3at^2$, by a previous example. The average acceleration over the interval $t'-t$ is

$$\frac{\dot{x}'-\dot{x}}{t'-t}=\frac{3at'^2-3at^2}{t'-t}=3a(t'+t),$$

and the result of making $t'\to t$ is clearly $\ddot{x}=f''(t)=6at$.

43. In the notation of the differential calculus,

$$\dot{x}=\text{Lim.}\frac{\delta x}{\delta t}=\frac{dx}{dt}$$

as explained in § 39. In the same way,

$$\ddot{x} = \text{Lim.}\, \frac{\delta \dot{x}}{\delta t} = \frac{d\dot{x}}{dt}.$$

In each case the symbol on the right stands for the result of a certain process or operation performed on x (or $\dot{x}$) when expressed in terms of t. Thus the second equality may be written

$$\ddot{x} = \frac{d}{dt}(\dot{x}) = \frac{d}{dt}\left[\frac{d}{dt}(x)\right],$$

and this again more succinctly in the usual notation

$$\ddot{x} = \frac{d^2x}{dt^2} = f''(t).$$

This adds nothing beyond a convenient notation to what has been previously explained.

Another expression for the acceleration is sometimes valuable. It has been assumed that x and $\dot{x}$ are expressed in terms of t. Alternatively the velocity $\dot{x}$ may be expressed in terms of x without reference to t. Now

$$\ddot{x} = \text{Lim.}\, \frac{\delta \dot{x}}{\delta t} = \text{Lim.}\, \frac{\delta x}{\delta t} \cdot \frac{\delta \dot{x}}{\delta x}$$

$$= \text{Lim.}\, \frac{\delta x}{\delta t} \,.\, \text{Lim.}\, \frac{\delta \dot{x}}{\delta x} = \dot{x}\frac{d\dot{x}}{dx};$$

and therefore, when the velocity is so expressed in terms of x, the acceleration is readily obtained in the same form. Thus, if $\dot{x} = f(x)$,

$$\ddot{x} = f(x)\,.\,f'(x),$$

where $f'(x)$ is the derivative of $f(x)$.

Ex. If the velocity varies as x, so does the acceleration.

Ex. A graph is drawn by plotting the velocity $\dot{x}$ as ordinate against

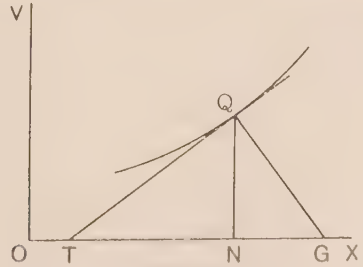

FIG. 26.

x. Let QN, QT, QG be the ordinate, tangent and normal to the curve at Q $(x, \dot{x})$, so that

$$\dot{x} = QN, \quad \frac{d\dot{x}}{dx} = \tan QTN = \tan GQN.$$

Therefore the acceleration is

$$\ddot{x} = QN \tan GQN = NG,$$

and is represented graphically by the *sub-normal*, as the projection of the normal QG on OX is called.

44. Certain particular types of motion will now be examined in detail. The equation $x = a$, not involving t, asserts that the point P occupies a stationary position. The equation $x = a + bt$ implies that the point P moves with uniform velocity b. The next equation in order of simplicity is naturally

$$x = a + bt + ct^2,$$

and this form can now be discussed.

The average velocity in the interval from t to t' is

$$\frac{x' - x}{t' - t} = \frac{(a + bt' + ct'^2) - (a + bt + ct^2)}{t' - t}$$

$$= \frac{b(t' - t) + c(t'^2 - t^2)}{t' - t}$$

$$= b + c(t' + t).$$

The actual velocity at the instant $t\,(t' \to t)$ is therefore

$$\dot{x} = b + 2ct.$$

The acceleration is

$$\frac{\dot{x}' - \dot{x}}{t' - t} = \frac{(b + 2ct') - (b + 2ct)}{t' - t}$$

$$= 2c,$$

and is therefore constant or uniform.

45. The conditions of this problem will now be reversed. Instead of deducing the uniform acceleration of P from an assumed expression for x in terms of t, the acceleration f is given and the position and velocity of P are to be found.

Since

$$f = 2c, \text{ or } c = \tfrac{1}{2}f,$$

$$x = a + bt + \tfrac{1}{2}ft^2,$$

$$\dot{x} = b + ft;$$

and if f alone is given, a and b are any two constants whatever. Two such constants must always occur when the acceleration only

is known, for the motion of the point can always satisfy two further conditions. To take these conditions here in the simplest form, at any given time, say $t=0$, the point can start from any given position ($x=a$) with any given velocity ($\dot{x}=b$).

Let u be the initial velocity ($t=0$), v the velocity at time t, and s the distance traversed in the time t, so that $b=u$, $\dot{x}=v$ and $x-a=s$. Then

$$s=ut+\tfrac{1}{2}ft^2, \quad \ldots\ldots(7)$$

$$v=u+ft, \quad \ldots\ldots(8)$$

are the fundamental equations in this type of motion. The result of eliminating t,

$$fs=\tfrac{1}{2}ft(2u+ft)$$

$$=\tfrac{1}{2}(v^2-u^2), \quad \ldots\ldots(9)$$

is equally important. The equations

$$s=vt-\tfrac{1}{2}ft^2=\tfrac{1}{2}(u+v)t$$

are also simple deductions.

When the moving point starts from rest, $u=0$ and

$$v=ft, \quad s=\tfrac{1}{2}ft^2=\tfrac{1}{2}vt, \quad fs=\tfrac{1}{2}v^2,$$

whence

$$v=\sqrt{(2fs)}, \quad t=\sqrt{(2s/f)}.$$

But among all these numerous relations it is only essential to remember (7) and (8), from which the rest can be derived.

46. At any place on the surface of the earth a body falling freely, without resistance by the air, has a uniform acceleration vertically downwards. This acceleration, which is the same for all bodies, is commonly denoted by g. The value of g varies slightly, according to the locality, from the equator to the poles, where it is greatest. But $g=32{\cdot}2$ ft./sec.^{2} may be taken as a general value.

Ex. In what time will a point move a distance of 16 yards with uniform retardation 5 ft./sec.^{2}, the velocity at the middle point being 17 ft./sec.

Here $f=-5$ and the half distance is $s=24$ (ft.). Let u be the initial and v the final velocity. By (9)

$$-120=\tfrac{1}{2}(289-u^2)=\tfrac{1}{2}(v^2-289),$$

$$u^2=529, \quad v^2=49, \quad u=23, \quad v=7.$$

Hence the time over $2s$ is

$$t=\frac{2s}{\frac{1}{2}(u+v)}=\frac{48}{15}=3{\cdot}2 \text{ secs.}$$

Ex. A juggler keeps five balls continually in the air, throwing each to a height of 10 ft. What is the interval between each throw, and at

what height are the other balls at the moment when one reaches his hand ?

In more general terms, let h be the height. Then the velocity of projection is $u=\sqrt{(2gh)}$ and the time up or down is $\sqrt{(2h/g)}$. If n balls are kept in constant circulation, the interval between throws is therefore $2\sqrt{(2h/g)}/n$. The r-th preceding ball has been in flight a time $t=2r\sqrt{(2h/g)}/n$ at the moment when one comes to hand, and will have reached a height

$$\begin{aligned} s &= ut-\tfrac{1}{2}gt^2 \\ &= \sqrt{(2gh)}\,.\,2r\sqrt{(2h/g)}/n-2r^2\,.\,2h/n^2 \\ &= 4rh/n-4r^2h/n^2 \\ &= 4h\,.\,r(n-r)/n^2. \end{aligned}$$

When $h=10$ ft. and $n=5$, the height of the r-th preceding ball is $1{\cdot}6r(5-r)$, giving, for the ascending balls ($r=1$, 2) heights of 6·4 and 9·6 ft., and the same in reverse order for the descending balls ($r=3$, 4). The interval between successive throws is

$$0{\cdot}4\sqrt{(20/32{\cdot}2)}=\sqrt{(32/322)}=0{\cdot}32 \text{ sec.}$$

47. The second type of motion which merits discussion is represented by the equation

$$x=a\sin(nt-\alpha),$$

where the angle $nt-\alpha$ is taken to be expressed in circular measure.

When	$t=$	α/n,	$(\frac{1}{2}\pi+\alpha)/n$,	$(\pi+\alpha)/n$,	$(\frac{3}{2}\pi+\alpha)/n$,
	$x=$	0,	$+a$,	0,	$-a$,

and the displacements are repeated in the same order every time that t is increased by $2\pi/n$. The equation represents the simplest type of periodic motion, the period of oscillation being $T=2\pi/n$. This type is known as *simple harmonic motion*. The range of displacement is between $+a$ and $-a$, and a is called the *amplitude*.

The average velocity in the interval from t to t' is

$$\begin{aligned} \frac{x'-x}{t'-t} &= a\,.\,\frac{\sin(nt'-\alpha)-\sin(nt-\alpha)}{t'-t} \\ &= 2a\,.\,\frac{\sin\frac{1}{2}n(t'-t)}{t'-t}\,.\,\cos\{\tfrac{1}{2}n(t'+t)-\alpha\}. \end{aligned}$$

But, the angle being expressed in circular measure,

$$\frac{\sin\frac{1}{2}n(t'-t)}{\frac{1}{2}n(t'-t)}\to 1, \quad (t'\to t).$$

Hence the velocity at the instant t is

$$\dot{x}=\text{Lim.}\,\frac{x'-x}{t'-t}=an\cos(nt-\alpha).$$

The same process may be repeated to deduce the acceleration $\ddot{x}$ from $\dot{x}$. Or if it is noticed that the result of the process is to derive from $x=a\sin(nt-\alpha)$,

$$\dot{x}=a\,.\,n\sin(nt-\alpha+\tfrac{1}{2}\pi),$$

it is clear that a repetition of it must lead to

$$\ddot{x}=an\,.\,n\sin(nt-\alpha+\tfrac{1}{2}\pi+\tfrac{1}{2}\pi)$$
$$=-an^2\sin(nt-\alpha).$$

Naturally, since the displacement is periodic, the velocity and acceleration are periodic also. By eliminating t they may also be expressed in the forms

$$\dot{x}=\pm n\sqrt{(a^2-x^2)},$$
$$\ddot{x}=\pm n\sqrt{(a^2n^2-\dot{x}^2)}=-n^2x.$$

The last is most important. It shows that the acceleration varies as the displacement, and, being of opposite sign, is always directed in the opposite sense, that is, towards the fixed origin.

48. This is another illustration of the fact that when a relation is given between the displacement and the time, in the form $x=f(t)$, it is comparatively easy, by a direct process, to deduce the velocity and acceleration of the moving point at any instant. The technique of the calculus will make the process simpler in general, but it is not indispensable in treating simple cases. On the other hand, the process cannot be reversed with equal facility; when the acceleration, or even the velocity, is given, it may not be easy, and in a restricted sense it may even be impossible, to deduce an expression for the displacement in terms of the time. The difficulty will probably seem still more apparent when the acceleration is given in terms, not of the time, but of the displacement.

Thus in the present case

$$\ddot{x}=\frac{d^2x}{dt^2}=-n^2x;$$

and although this is the simplest equation of its type, as well as an extremely important one, it will be evident that the steps by which it was arrived at cannot be simply retraced. In the absence of an independent method, advantage must be taken of what has already been learnt, which shows that the type of motion defined by

$$x=a\sin(nt-\alpha)$$

leads to the assigned acceleration. Moreover, this is the most general equation possible, because the two independent constants,

a and α, can be so chosen as to satisfy any required conditions. It is also known that the equation leads to the velocity,

$$\dot{x} = an \cos(nt - \alpha).$$

Hence if at the time $t = 0$ the point $x = x_1$ is passed with the velocity $\dot{x} = u$,

$$x_1 = -a \sin \alpha, \quad u/n = a \cos \alpha,$$

in solving which it is convenient to make the amplitude a positive, and to fix the quadrant of the *phase* α accordingly. Other given conditions can be satisfied in a similar way.

Ex. In the course of a single vibration in S.H.M. the displacements $x = 1, 4, 3$ (ft.) are observed to occur at the times $t = 1, 2, 3$ (secs.) respectively. To find the amplitude and period.

According to the data,

$$1 = a \sin(n - \alpha), \quad 4 = a \sin(2n - \alpha), \quad 3 = a \sin(3n - \alpha),$$

and therefore

$$\begin{aligned} 4 = a \sin(2n - \alpha) &= a\{\sin(n - \alpha) + \sin(3n - \alpha)\} \\ &= 2a \sin(2n - \alpha) \cos n, \end{aligned}$$

or $\cos n = \frac{1}{2}$, $n = \frac{1}{3}\pi$, and the period is $2\pi/n = 6$ secs.

Again,

$$\begin{aligned} 2 &= a\{\sin(3n - \alpha) - \sin(n - \alpha)\} \\ &= 2a \sin n \cos(2n - \alpha); \end{aligned}$$

or since $\sin n = \sqrt{3}/2$,

$$2/\sqrt{3} = a \cos(2n - \alpha), \quad 4 = a \sin(2n - \alpha),$$

$$a^2 = 16 + \tfrac{4}{3}, \quad a = \sqrt{(\tfrac{52}{3})} = 4{\cdot}16 \text{ ft.}$$

Also $\sin(3n - \alpha) = \sin \alpha = 3/a = \sqrt{(\frac{9}{52})}$, so that α is a little greater than $\frac{1}{4}\pi$.

Ex. From $\dot{x} = n\sqrt{(a^2 - x^2)}$, deduce $\ddot{x} = -n^2 x$ directly. If $\dot{x}$ changes to $\dot{x}'$ when x changes to x',

$$\dot{x}^2 = n^2(a^2 - x^2), \quad \dot{x}'^2 = n^2(a^2 - x'^2),$$

$$\dot{x}'^2 - \dot{x}^2 = -n^2(x'^2 - x^2),$$

$$\frac{d\dot{x}}{dx} = \text{Lim.}\, \frac{\dot{x}' - \dot{x}}{x' - x} = -n^2 . \text{Lim.}\, \frac{x' + x}{\dot{x}' + \dot{x}} = -\frac{n^2 x}{\dot{x}}.$$

Hence

$$\ddot{x} = \dot{x} \frac{d\dot{x}}{dx} = -n^2 x.$$

49. The problem of finding the displacement x in the form $f(t)$ when the velocity $\dot{x}$ is given in the form $f'(t)$ will now be considered from another point of view.

The velocity $\dot{x}$ can be plotted as the ordinates of a graph against the time t as abscissa. The displacement is required in the interval

from t_0 to $t=t_n$. Let this interval be divided into n equal intervals at $t_1, t_2, \ldots, t_{n-1}$, so that $t_1-t_0=t_2-t_1=\ldots=t_n-t_{n-1}=(t_n-t_0)/n$. The corresponding ordinates of the graph are

$$P_0t_0=\dot{x}_0, \quad P_1t_1=\dot{x}_1, \ldots, P_nt_n=\dot{x}_n.$$

Complete the rectangles $P_0M_1P_1N_1$, $P_1M_2P_2N_2, \ldots, P_{n-1}M_nP_nN_n$ as in the figure. Now the area $P_0t_0t_1M_1=\dot{x}_0(t_1-t_0)$, or the displace-

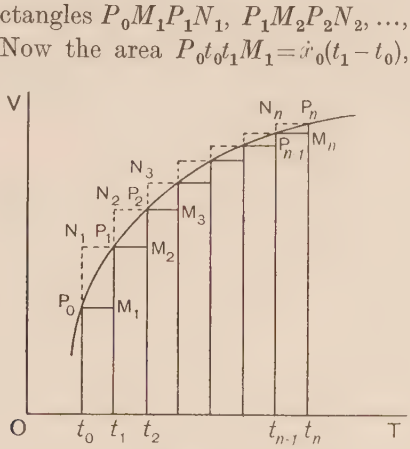

FIG. 27.

ment which would follow from a uniform velocity $\dot{x}_0$ maintained through the interval (t_1-t_0). The actual displacement is greater, because the average velocity in the interval is clearly greater. Similarly the area $N_1t_0t_1P_1=\dot{x}_1(t_1-t_0)$, which is greater than the actual displacement because the average velocity in this interval is clearly less than $\dot{x}_1$.

Let $x_0, x_1, \ldots, x_n$ be the displacements at the times $t_0, t_1, \ldots, t_n$. By what has already been seen,

$$\text{area } P_0t_0t_1M_1 < x_1-x_0 < \text{area } N_1t_0t_1P_1,$$

and similarly

$$\text{area } P_1t_1t_2M_2 < x_2-x_1 < \text{area } N_2t_1t_2P_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\text{area } P_{n-1}t_{n-1}t_nM_n < x_n-x_{n-1} < \text{area } N_nt_{n-1}t_nP_n.$$

Hence, by addition,

$$\text{area } P_0M_1P_1M_2P_2\ldots M_nt_nt_0 < x_n-x_0 < \text{area } P_0N_1P_1N_2P_2\ldots P_nt_nt_0.$$

Now the difference of these areas is the sum of the rectangles P_0P_1, $P_1P_2, \ldots, P_{n-1}P_n$, and is

$$\begin{aligned}&(\dot{x}_1-\dot{x}_0)(t_1-t_0)+(\dot{x}_2-\dot{x}_1)(t_2-t_1)+\ldots+(\dot{x}_n-\dot{x}_{n-1})(t_n-t_{n-1})\\&=(\dot{x}_1-\dot{x}_0+\dot{x}_2-\dot{x}_1+\ldots+\dot{x}_n-\dot{x}_{n-1})(t_n-t_0)/n\\&=(\dot{x}_n-\dot{x}_0)(t_n-t_0)/n,\end{aligned}$$

which can be made less than any assignable quantity by making n correspondingly large. Hence, when n is made indefinitely large, both areas converge to the same limit, which is the area contained between the velocity curve, the axis of t, and the extreme ordinates. This area therefore represents the required displacement.

Ex. In the case of uniform acceleration, $\dot{x} = u + ft$ and the velocity graph is a straight line. The ordinate at t being $u + ft$, the ordinate at O is u, and the displacement s in the interval 0 to t is represented by the area of a trapezium, so that

$$s = \tfrac{1}{2}t(u + u + ft) = ut + \tfrac{1}{2}ft^2.$$

Ex. If $\dot{x} = \sqrt{(a^2 - t^2)}$, or $\dot{x}^2 + t^2 = a^2$, the graph of the velocity is a

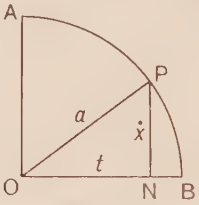

FIG. 28.

circle of radius a. The displacement in the interval 0 to t is represented by the area

$$OAPN = OAB - OPB + OPN$$
$$= \frac{1}{4}\pi a^2 - \frac{1}{2}a^2 \cos^{-1}\frac{t}{a} + \frac{1}{2}t\sqrt{(a^2 - t^2)}.$$

Hence the displacement is

$$s = \frac{1}{2}a^2 \sin^{-1}\frac{t}{a} + \frac{1}{2}t\sqrt{(a^2 - t^2)}.$$

50. It is now clear that the problem of deducing the displacement of a point in any interval of time, when the relation of velocity to time is known, is essentially the same as that of finding the area of a certain curve, the time-velocity graph. This in itself does not advance the solution, unless the graph is a particularly simple curve.

To go further, let t_r, t_{r+1} be two consecutive instants at which the interval t_0 to t_n has been divided, and x_r, x_{r+1} the corresponding displacements, $\dot{x}_r$, $\dot{x}_{r+1}$ the corresponding velocities. The typical inequality connecting the displacement $x_{r+1} - x_r$ with the areas of the corresponding rectangular strips can be written

$$\dot{x}_r(t_{r+1} - t_r) < x_{r+1} - x_r < \dot{x}_{r+1}(t_{r+1} - t_r),$$

and it is convenient to write the sum of all such inequalities in the form

$$\Sigma\dot{x}_r(t_{r+1} - t_r) < x_n - x_0 < \Sigma\dot{x}_{r+1}(t_{r+1} - t_r),$$

where Σ before an expression implies the sum of all such expressions obtained by giving r all the values $0, 1, \ldots, n-1$ in succession. Further, it has been seen that the limit of the two sums Σ is the same when n is increased indefinitely. Hence it is possible to write

$$\begin{aligned} x_n - x_0 &= \text{Lim. } \Sigma \dot{x}_r (t_{r+1} - t_r) \\ &= \text{Lim. } \Sigma \dot{x}_{r+1} (t_{r+1} - t_r), \quad (n \to \infty). \end{aligned}$$

Now this result is expressed in the notation of the Integral Calculus in the form of a *definite integral* thus :

$$x_n - x_0 = \int_{t_0}^{t_n} \dot{x}\, dt,$$

where the symbol $\int$ (like Σ a variant of S) denotes summation and at the same time implies that the interval $t_n - t_0$ has been indefinitely sub-divided. The same expression represents the area of the time-velocity graph.

51. A simple case will illustrate this process of summation, and the result will be of practical use later. Let $\dot{x} = t^p$, where p is a positive integer. Since the sums $\Sigma \dot{x}_r (t_{r+1} - t_r)$ and $\Sigma \dot{x}_{r+1} (t_{r+1} - t_r)$ converge to the same limit, that limit can equally be found by replacing $\dot{x}_r = t_r^{\,p}$ or $\dot{x}_{r+1} = t_{r+1}^{\,p}$ by any intermediate quantity.

Now $(p+1) t_r^{\,p} < t_r^{\,p} + t_r^{\,p-1} t_{r+1} + \ldots + t_{r+1}^{\,p} < (p+1) t_{r+1}^{\,p}$,

since there are $(p+1)$ terms in the sum, where each term bears to the preceding the ratio t_{r+1}/t_r. Hence

$$\begin{aligned} x_n - x_0 &= \int_{t_0}^{t_n} t^p\, dt \\ &= \text{Lim. } \Sigma \frac{1}{p+1} (t_r^{\,p} + t_r^{\,p-1} t_{r+1} + \ldots + t_{r+1}^{\,p}) (t_{r+1} - t_r) \\ &= \frac{1}{p+1} \text{ Lim. } \Sigma (t_{r+1}^{\,p+1} - t_r^{\,p+1}) \\ &= \frac{1}{p+1} (t_n^{\,p+1} - t_{n-1}^{\,p+1} + t_{n-1}^{\,p+1} - t_{n-2}^{\,p+1} + \ldots + t_1^{\,p+1} - t_0^{\,p+1}) \\ &= \frac{1}{p+1} (t_n^{\,p+1} - t_0^{\,p+1}). \end{aligned}$$

The limit has become superfluous in this case, because it was easy to choose a quantity intermediate between $t_r^{\,p}$ and $t_{r+1}^{\,p}$ in such a way as to make the process of summation independent of the number of terms n into which the interval $t_n - t_0$ was divided. As

this is not usually feasible, the method must be regarded as illustrative, and not as one of general utility.

The result is true for all values of p, except $p=-1$, when it is plainly illusory. The more general proof is not required, and will not be given here.

52. It will be noticed that in § 49, according to the figure given, the velocity $\dot{x}$ has been assumed to be continuously increasing through the interval and positive. Had it been always decreasing it is easy to see that the inequalities would be reversed, but the result of the reasoning would be unaffected. More generally, the interval can be divided into a certain number of parts, in each of which the velocity either increases or decreases, and the result established for each remains true for the total extent of the interval. Again, when the velocity is negative, the ordinates of the graph are below the axis of time. An area below this axis therefore represents a negative displacement and must be taken in the negative sense.

These are simple matters of form and interpretation. It is unnecessary, and it would be out of place, to enter into those more difficult and subtle questions which inevitably arise in the abstract theory of the calculus. But the opportunity cannot be missed of bringing such results as have been obtained into due relation, for this will explain in the simplest way the need for an appropriate calculus and the nature of its fundamental ideas.

53. In expressing the displacement by a definite integral, the limits (t_0, t_n) between which the integration (or summation) is to be effected are naturally exhibited. But the mere relation between x and t is equally represented in the form of an *indefinite integral* thus :

$$x=\int \dot{x}\, dt+C.$$

Here the upper limit, taking the place of t_n, is understood to be t. Neither the lower limit t_0 nor the corresponding value $x=x_0$ is expressed, but the addition of the arbitrary constant C permits of an equivalent adjustment of corresponding values of x and t. The integral thus represents a function of t, indefinite to the extent of an additive, adjustable constant.

But on the other hand, it has been seen that

$$x=f(t), \quad \dot{x}=\frac{dx}{dt}=f'(t),$$

where $f'(t)$ is the derivative of $f(t)$. Comparison with the integral form shows immediately that

$$x=\int\left(\frac{dx}{dt}\right)dt+C$$

and

$$\dot{x}=\frac{dx}{dt}=\frac{d}{dt}\left[\int \dot{x}\,dt\right]$$

the constant C naturally disappearing on being differentiated.

It is therefore clearly to be inferred that the symbols $\int(\,)\,dt$ of *integration* and $\frac{d}{dt}$ of *differentiation* represent inverse operations applied to a function, the effect of either one being to undo the effect of the other when both are performed in succession.

54. The result of substituting $x=f(t)$, $\dot{x}=f'(t)$ in the integral form is

$$f(t)=\int f'(t)\,dt+C,$$

or in words, when the *integrand* $f'(t)$ is given, the integral (with an additive arbitrary constant understood) is that function of which the integrand is the derivative. A knowledge of the derivatives of a certain number of standard functions, supplemented by various methods of transformation and reduction, gives in all favourable cases a more practical mode of calculating integrals than that furnished by direct summation.

Without entering into further detail, the material for useful illustration has already been obtained. In § 47 it has been shown that in one problem,

$$x=a\sin(nt-\alpha),\quad \dot{x}=an\cos(nt-\alpha).$$

Hence $(a=1)$ $$\sin(nt-\alpha)=n\int\cos(nt-\alpha)\,dt+C,$$

including as particular cases $(\alpha=0,\ \frac{1}{2}\pi)$

$$\int\cos nt\,dt=\frac{1}{n}\sin nt+c,$$

$$\int\sin nt\,dt=-\frac{1}{n}\cos nt+c,$$

where a clearly permissible change has been made in the arbitrary constant.

To go in the opposite direction, the result of § 51 gives

$$\int t^p\,dt = t^{p+1}/(p+1)+c,$$

whence

$$t^p = \frac{1}{p+1}\cdot\frac{d}{dt}(t^{p+1});$$

or, on changing $p+1$ to q,

$$\frac{d}{dt}(t^q) = qt^{q-1},$$

which is true generally, though proved only for $q-1$ a positive integer.

55. The connexion between the indefinite integral and the derivative being now explained, the relation between the indefinite and the definite integral may perhaps be seen more clearly. Since

$$x = f(t) = \int \dot{x}\,dt + C,$$

if x_0, x_1 are values of x corresponding to $t=t_0$, $t=t_1$,

$$x_0 = f(t_0) = \int^{t_0} \dot{x}\,dt + C,$$

$$x_1 = f(t_1) = \int^{t_1} \dot{x}\,dt + C,$$

where the upper limits are expressed, and the lower limits, in a given motion, are the same in both cases. Hence

$$x_1 - x_0 = f(t_1) - f(t_0) = \int^{t_1} \dot{x}\,dt - \int^{t_0} \dot{x}\,dt$$

$$= \int_{t_0}^{t_1} \dot{x}\,dt,$$

and the meaning of these integrals should be easily understood when it is remembered that they represent the area of the velocity graph between certain ordinates.

The definite integral is derived from the indefinite integral by a process in the course of which the arbitrary constant C disappears. This process of substitution and subtraction can be conveniently expressed by the notation

$$\int_{t_0}^{t_1} f'(t)dt = f(t_1) - f(t_0) = \Big[\, f(t) \Big]_{t_0}^{t_1}.$$

So much can be learnt without passing beyond the threshold of the Integral Calculus. At the same time, a certain reserve is necessary. The above relations have been found very simply by a

consideration of the area of the velocity graph, and in doing so it has been assumed that this graph is a perfectly simple curve, free from complications of any kind. In nature the velocity of a point changes continuously and is never infinite. If, however, the integrand $f'(t)$ is an abstract function, unrestricted in its properties, the above intuitive method ceases to be adequate.

56. Since acceleration bears to velocity the same relation as velocity to displacement, the displacement can be derived when the acceleration is given in terms of the time, by the application of the same method in two successive stages.

In the first stage,

$$\ddot{x}=\frac{d\dot{x}}{dt}, \quad \dot{x}=\int \ddot{x}\,dt+C_1$$

are equivalent statements. But

$$x=\int \dot{x}\,dt+C_2.$$

Hence

$$x=\int\left[\int \ddot{x}\,dt\right]dt+\int C_1\,dt+C_2$$

$$=\iint \ddot{x}\,dt^2+C_1t+C_2,$$

where the result of the repeated integration of $\ddot{x}$ is represented by the usual symbol for a double integral.

Thus, when the acceleration $\ddot{x}$ is given in terms of the time, two distinct arbitrary constants, C_1, C_2, occur in the expression of the displacement x. These, as examples have shown, are always necessary in order to express the two conditions which must be given in every problem of this type, such as the position and velocity of the moving point at a stated time.

57. Another expression for the acceleration, which has been found, is

$$\ddot{x}=\dot{x}\frac{d\dot{x}}{dx}.$$

Now

$$\frac{d}{dx}(\dot{x}^2)=\text{Lim.}\,\frac{\dot{x}'^2-\dot{x}^2}{x'-x}, \quad (x'\to x)$$

$$=\text{Lim.}\,(\dot{x}'+\dot{x})\cdot\frac{\dot{x}'-\dot{x}}{x'-x}$$

$$=2\dot{x}\frac{d\dot{x}}{dx},$$

and therefore

$$2\ddot{x}=\frac{d}{dx}(\dot{x}^2).$$

Hence, when this result is expressed in the equivalent integral form,

$$\dot{x}^2 = 2\int \ddot{x}\, dx + C,$$

which gives the velocity when the acceleration is expressed in terms of x.

Ex. When the acceleration is uniform, $\ddot{x} = f$, and

$$\dot{x}^2 = 2fx + C.$$

Ex. When the acceleration is given by $\ddot{x} = -n^2x$,

$$\dot{x}^2 = -2n^2\int x\, dx + C$$
$$= -n^2x^2 + C,$$

and therefore, on writing n^2a^2 for C,

$$\dot{x} = n\sqrt{(a^2 - x^2)}$$

a result already obtained in this case.

58. In what precedes, the motion of a point P along a straight line has been considered, x being the displacement of P, from a fixed point O on the line, at the time t. The velocity and acceleration of P are necessarily along the line, the two possible directions along the line being distinguished in the usual way by the sign attached (+ or −) to either quantity. The motion, being confined to the straight line, is thus completely defined.

But it will be observed that the reasoning employed rests entirely on the mode of traversing the line, and not at all on the shape of the line. If the straight line were bent (without stretching) into the form of a curve, not necessarily a plane curve, sections of the straight line would be bent into arcs of equal length, and all the ideas depending on the rate of description of the straight line could be applied immediately to the motion of a point describing the curve. In fact, if s represents the length of arc measured on the curve from a fixed point O to a point P moving on it, equations such as

$$\dot{s} = \frac{ds}{dt}, \quad s = \int \dot{s}\, dt,$$
$$\ddot{s} = \frac{d\dot{s}}{dt} = \frac{d^2s}{dt^2} = \dot{s}\frac{d\dot{s}}{ds}$$
$$\dot{s} = \int \ddot{s}\, dt, \quad s = \iint \ddot{s}\, dt, \quad \tfrac{1}{2}\dot{s}^2 = \int \ddot{s}\, ds,$$

all have a meaning obviously suggested by analogy. They can be accepted as true without further examination when the curve, along

which the point is moving, is treated as a one-dimensional medium, with the abstraction of all its relations to space.

This, of course, is an inadequate view. The symbol $\dot{s}$, for example, clearly stands for the rate of motion along the curve at the instant t. But this motion is variable not only in the magnitude of its rate but also in direction. The latter element is an essential part of the velocity, and in its absence the magnitude of the velocity may conveniently be designated by the word *speed*. The symbol $\ddot{s}$ presents a greater difficulty. Superficially it stands for the rate at which the speed of the moving point is changing at the instant t. But what relation it bears to the true acceleration of the moving point is a question which cannot be answered by intuitive considerations. The wider meaning of the above equations will therefore require later examination.

59. The idea represented by the rate of change of a variable quantity, and also the conventional mode of expressing it, should now be familiar. When the variable is some geometrical quantity used as a coordinate, its rate of change with the time is commonly called a velocity, distinguished by some qualification, such as angular or areal. Similarly, the rate of change of such a velocity may be called an acceleration, with the same qualification.

If O is a fixed point, OX a fixed line, and P a point moving in a plane containing OX, the angle $POX=\theta$ varies with the time. The *angular velocity* of OP or of P is the rate of change of this angle, and is

$$\dot{\theta}=\frac{d\theta}{dt},$$

while the *angular acceleration* of OP or of P is similarly

$$\ddot{\theta}=\frac{d\dot{\theta}}{dt}=\frac{d^2\theta}{dt^2}=\dot{\theta}\frac{d\dot{\theta}}{d\theta}.$$

If in this case $OP=r$ varies with the time, the motion of P along OP can be considered quite irrespective of the fact that OP is itself rotating. The rate of change of r is the *radial velocity* of P, and is

$$\dot{r}=\frac{dr}{dt},$$

while the acceleration of P along the *moving* radius

$$\ddot{r}=\frac{d\dot{r}}{dt}=\frac{d^2r}{dt^2}=\dot{r}\frac{d\dot{r}}{dr}.$$

These equations are no more than definitions, symbolically expressed, to be further examined in the next chapter.

60. The simplest type of motion which is naturally expressed in these terms is clearly presented by the case in which P describes a circle uniformly about O, for then $r=a$ and $\dot{\theta}=n$ are both constants. In this connexion it will be found convenient to express angles in circular measure, so that the unit of angular velocity is one radian ($57°{\cdot}3$) per unit of time. If $\theta=0$ when $t=0$, $\theta=nt$ generally. But the speed of P in its path, $\dot{s}=v$, is also constant, so that $s=vt$, if $s=0$ at the instant $t=0$. Let T be the period in which the circle is described. Then

$$2\pi=nT, \quad 2\pi a=vT,$$

and therefore

$$T=2\pi/n, \quad v=na.$$

It is clear that the speed is always along the tangent and at right angles to the radius OP.

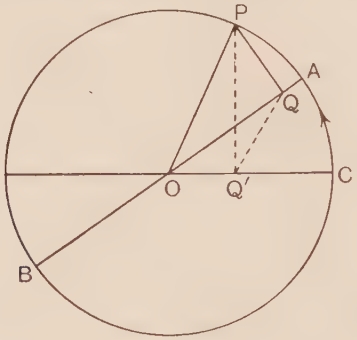

FIG. 29.

At the time $t=0$ let C be the position of the moving point. At any time t the arc $CP=s=vt$ and the angle $COP=\theta=nt$. Draw the diameter BOA, making the angle $COA=\alpha$.

Let Q be the foot of the perpendicular from P on this diameter. If $OQ=x$, since $AOP=\theta-\alpha$,

$$x=OP\cos AOP=a\cos(nt-\alpha).$$

But this is the relation which defines the linear motion considered above and described as simple harmonic. Hence simple harmonic motion is the motion of a point which is the projection on any line of a point describing a circle uniformly. The amplitude of the motion is clearly equal to the radius of the circle and the period is the same as that in the circular motion.

Ex. If Q' is the projection of P on OC, prove that Q, relatively to Q', describes a circle uniformly.

Since OQP, $OQ'P$ are right angles, Q, Q' lie on the circle on OP as diameter, and therefore $QQ'P = QOP = nt - \alpha$ and $OQQ' = OPQ'$. Hence $Q'Q = OQ' \sin QOQ' / \sin OQQ' = OQ' \sin \alpha / \sin OPQ' = a \sin \alpha$, which is constant. Also $Q'Q$ makes with PQ', a fixed direction, the angle $nt - \alpha$, which increases uniformly. Hence, relatively to Q', Q moves in a circle of radius $a \sin \alpha$ with uniform angular velocity n, but in the opposite direction to the motion of P.

EXAMPLES.

1. In uniform acceleration from A to B show that the average velocity is equal to the velocity midway in time and less than the velocity at the point midway in distance.

Illustrate the circumstances by a graph of $4x = t^2$.

2. A point starts from rest with uniform acceleration until its velocity is v. At this velocity it moves for a time t, and then comes to rest with uniform retardation. If T is the whole time occupied, find the distance travelled.

3. A train has a maximum speed of 50 miles an hour, which it gains by uniform acceleration from rest in 10 minutes or loses by uniform retardation in 5 minutes. Compared with a non-stop run of 100 miles, how much travelling time is lost by stopping at two stations 30 miles from each end? If the train stops every four miles, how long will it be in motion on the same journey?

4. A point P starts from O with uniform velocity 20 ft./sec. Ten seconds later a point Q starts from rest at O with uniform acceleration 3 ft./sec.^{2} in the same direction. When and where will Q overtake P?

Interpret clearly the alternative solution, and draw a graph by plotting the displacement of P and Q against the time, representing 4 secs. and 100 ft. by the same unit.

5. In uniform acceleration the points A, B, C on a straight line are passed with the velocities v_1, v_2, v_3. Prove that (with due regard to sign)

$$v_1^2 \cdot BC + v_2^2 \cdot CA + v_3^2 \cdot AB = 0.$$

6. Successive equal distances are passed over in times t_1, t_2, t_3, the acceleration being uniform. Prove that

$$(t_1 + t_2 + t_3)(t_1^{-1} - t_2^{-1} + t_3^{-1}) = 3.$$

7. A point A starts from the point O on a straight line with velocity v_1 and uniform acceleration f_1. After an interval T a point B leaves O with velocity v_2 and uniform accleration f_2. Find the conditions that (a) B and A may pass one another once only, (b) that B may just reach A without passing it. The signs of v_1, v_2, f_1, f_2 may be any whatever.

8. Assuming that g is increased effectively by the resistance of the air to $g(1+f)$ in upward motion, and reduced to $g(1-f)$ in downward

motion, find the time a stone is in the air when projected upward with the velocity v.

Show that there is a value of f which will make the time the same as in unresisted motion, and find it.

9. A stone is dropped into an empty pit of depth h. If it is heard to strike the bottom a time t after it is let fall, prove that

$$h = \tfrac{1}{2}gt^2\left(1 - \frac{gt}{V}\right)$$

approximately, where V is the velocity of sound.

Find the depth to the nearest foot when $g = 32{\cdot}2$ ft./sec.^{2}, $V = 1100$ ft./sec. and $t = 4$ secs.

10. Water is descending from a nozzle in a steady vertical jet. Assuming that the velocity at any horizontal section of the jet is that due to a simple fall from a certain height above the nozzle, show that

$$r^4(h + a) = b,$$

where r is the radius of the jet at a depth h below the nozzle, and a and b are constants.

11. The motion of a point is defined by $x = a \sin nt$. At the moment when its velocity is greatest, what uniform retardation will bring it to rest (a) in the same distance, (b) in the same time, as in simple harmonic motion ?

12. The driving wheels of a locomotive are coupled by a rod parallel to the rails. When the velocity is uniform show that the vertical motion of the coupling rod is simple harmonic.

The driving wheels are 6 ft. in diameter and the coupling rod is carried by cranks at 15 in. from the axle. When the engine is running at 40 miles an hour, find the angular velocity of the wheels and the greatest velocity and acceleration of the coupling rod in the vertical direction.

13. A point L leaves A with velocity v and comes to rest at B. The distance AB is divided into n equal intervals at $P_1, P_2, \ldots, P_{n-1}$. At P_1 the moving point receives a certain retardation, which is increased by an equal amount at $P_2, \ldots, P_{n-1}$. Prove that the retardation of the point when coming to rest at B is independent of n, and the same as it would be if the motion were simple harmonic about A as centre.

Prove also that the velocity at P_r is

$$v\left[\frac{(n-r)(n+r-1)}{n(n-1)}\right]^{\frac{1}{2}}.$$

14. A straight rod makes n simple harmonic oscillations per minute in the direction of its length, and a bead makes $n+1$ similar oscillations of equal amplitude per minute along the rod, n being a considerable number. Show that relatively to a fixed point the motion of the bead may be represented as a simple harmonic oscillation with a period

intermediate between those of the rod and of the bead on the rod, and with an amplitude steadily growing from 0 to twice the equal amplitudes and shrinking to 0 again every minute.

15. In simple harmonic motion v_1, v_2, v_3, v_4 are the velocities at the points A, B, C, D. Prove that (with due regard to signs)

$$\frac{v_1^2}{AB \cdot AC \cdot AD} + \frac{v_2^2}{BA \cdot BC \cdot BD} + \frac{v_3^2}{CA \cdot CB \cdot CD} + \frac{v_4^2}{DA \cdot DB \cdot DC} = 0.$$

16. Find the velocity and acceleration when the displacement $x = t(t-3)^2$. When and where does the point come to rest, and what is the velocity when there is no acceleration? Draw a (t, x) graph from $t=0$ to $t=4$.

17. The distances of A, B and M midway between A and B from a fixed point O along a straight line are x, y and $\frac{1}{2}(x+y)$. Point out the fallacy in the following argument: a point P moving along the line will pass A, B and M with the velocities $\dot{x}$, $\dot{y}$ and $\frac{1}{2}(\dot{x}+\dot{y})$; hence, whatever the type of motion, the velocity of P when passing M will be the arithmetic mean of its velocities at A and B.

18. When the displacement and time are connected by $x = at^{\frac{1}{2}}$, find the velocity and acceleration.

In the motion from one point to another show that the average velocity is equal to the velocity at the point midway in distance and greater than the velocity at the middle of the interval of time.

Illustrate by a graph of $x = 2t^{\frac{1}{2}}$, and point out in what way the conditions are related to those in Ex. 1.

19. A point moving on a straight line leaves the point $x=0$ at the time $t=0$ and its velocity is $V(\cos nt - \cos 2nt)$. Find the acceleration and the displacement at any time. When does the point come to rest and when is the speed greatest?

Draw the time-displacement graph for one period.

20. When the motion in a straight line is given by a^t, prove that the velocity is Aa^t and the acceleration is A^2a^t, where A is a constant depending on the value of a.

If $A = f(a)$, prove that

$$f(ab) = f(a) + f(b).$$

21. Show that the length of a curve is

$$s = \int \rho \, d\psi,$$

ρ being the radius of curvature at the point where the tangent makes the angle ψ with a fixed direction.

Show that the length of an arc of the involute to a circle of radius a from the point where it leaves the circle is $\frac{1}{2}a\psi^2$.

22. When a circle rolls uniformly on the inside of another circle of double its radius, show that any point on the circumference of the rolling circle describes a simple harmonic motion.

23. Points P and Q start from a point A on a circle and describe the circumference uniformly in opposite directions, P with double the speed of Q. The line PQ meets AB, the diameter through A, in M. Show that the motion of the point M is simple harmonic about B, the amplitude being equal to the diameter of the circle and the period equal to the time in which Q describes the circle.

24. Two straight rods passing through A and B revolve uniformly about these points in the same direction, the one through B three times as fast as the one through A. The rods, starting in the same direction AB, intersect in P and N is the foot of the perpendicular from P on AB. Show that the motion of N is simple harmonic about the middle point of AB, with the amplitude equal to AB and the period equal to half the time of one revolution of the rod through A.

CHAPTER III

MOTION OF A POINT IN A PLANE

61. When a point P is moving in a plane, it has at any instant a certain speed in a certain direction. This speed and direction together define a directed quantity, which is the velocity of P. The velocity of a point therefore possesses the first properties of a vector, being a magnitude associated with a direction.

But, in the nature of the case, since the position of P in the plane can be defined by two coordinates, their rates of change will also define the velocity of P. Thus if (x, y) are the rectangular coordinates of P, $\dot{x}$ may be regarded as the velocity of the ordinate through P and $\dot{y}$ as the velocity of P along this ordinate. Or if (r, θ) are the polar coordinates of P, $\dot{r}$ is the velocity of P along the radius vector, which itself is rotating about a fixed point O with angular velocity $\dot{\theta}$. Hence P may be regarded as having two independent velocities simultaneously, and it is necessary to find their single resultant.

Let the point have two velocities u, v. The velocity u, if continuing uniformly for unit time, would produce the displacement

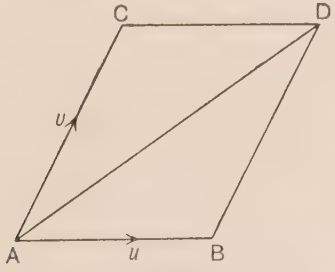

FIG. 30.

$AB = u$ units in its own direction. The velocity v similarly, if continuing uniformly for unit time, would produce the displacement $AC = v$ units in its own direction. The result of the two simultaneous

displacements is the single displacement $AD=R$ units, where AD is obtained by the parallelogram law. But this displacement would equally follow from a single velocity R if continued uniformly for unit time in the direction AD. Hence the resultant velocity is obtained, as regards both its magnitude and direction, from the component velocities according to the parallelogram law. The velocity of a point is therefore a vector, and can be resolved and compounded accordingly.

It will be noticed that, unless the component velocities are uniform, the finite displacements are hypothetical and not actual. This in no way affects the argument when the velocities are variable in magnitude and direction, for it is concerned with their values at a particular instant and not at all with what they may afterwards become. Otherwise it may be remarked that the unit of time employed in order to develop finite displacements is immaterial to the argument, and by making it sufficiently small the hypothetical displacements can be made to approximate to actual displacements as closely as may be desired.

In the second place, the argument is far wider than this particular application. What it shows is that linear velocity, because it is the rate of change of linear displacement, which is a vector, is also a vector. It will clearly remain true if any other kind of vector be substituted for the linear displacement. Indeed, no further proof is required for the quite general theorem that the rate of change of a vector is itself a vector.

Now the acceleration of a point is defined as the rate of change of its velocity. Since velocity is a vector, so is acceleration.

62. The addition of two vectors requires only the simplest trigonometry. Let u, v be the two vectors, making the angle α with one

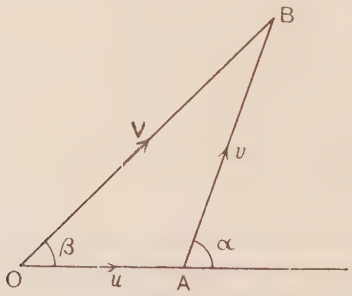

FIG. 31.

another : this angle is measured from the forward direction of the one to the forward direction of the other. Let V be the resultant, making the angle β with u. Then clearly

$$V^2 = u^2 + v^2 + 2uv\cos\alpha,$$

$$\sin\beta = v\sin\alpha/V,$$

from the relations in the triangle OAB.

When it is required to combine more than two vectors in one plane, a more systematic method is commonly useful. Let the vectors be u_1, u_2, u_3, ..., making the angles α_1, α_2, α_3, ..., with a

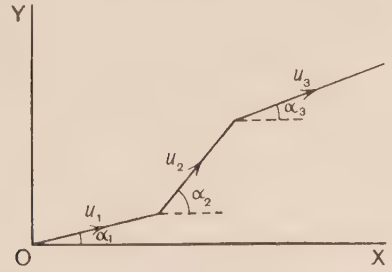

FIG. 32.

fixed direction OX. Take OY at right angles to OX. The components of the vectors parallel to OX are $u_1\cos\alpha_1$, $u_2\cos\alpha_2$, ..., and the components parallel to OY are $u_1\sin\alpha_1$, $u_2\sin\alpha_2$, ... Let V be the required resultant, making an angle β with OX, so that its components are $V\cos\beta$ parallel to OX, and $V\sin\beta$ parallel to OY. Then

$$V\cos\beta = u_1\cos\alpha_1 + u_2\cos\alpha_2 + \ldots,$$

$$V\sin\beta = u_1\sin\alpha_1 + u_2\sin\alpha_2 + \ldots,$$

and these equations can conveniently be written, for any number of components, in the form

$$V\cos\beta = \Sigma u\cos\alpha,$$

$$V\sin\beta = \Sigma u\sin\alpha,$$

where Σ stands for the sum of any number of terms similarly formed. In general V may be taken to be positive, and the quadrant of β is to be adjusted accordingly.

63. Consider once more the triangle OAB, in which the side OA represents the vector u, the side AB the vector v and the side OB the vector V. If these vectors are the displacements of a point, it is self-evident that V is the resultant of u and v. Because velocities are vectors it is equally true that a point endowed with the indepen-

dent velocities u and v possesses the resultant velocity V. Since accelerations are also vectors the same is true when the word velocity is replaced by the word acceleration.

But the same relations exhibited in this simple figure can also be interpreted in a quite different sense. Thus let $OA=u$ be the displacement of one point A and $OB=V$ the displacement of another point B. In the result $AB=v$ represents the displacement of B relative to A. Equally, in the case of velocities, $AB=v$ represents, in magnitude and direction, the velocity of B relative to A. Similarly, when accelerations are in question, $AB=v$ represents the acceleration of B relative to A.

This gives a natural meaning to the vectorial equation $V-u=v$. To take the case of velocities, any common motion assigned to A and B will leave their relative positions unaltered. Let this common motion be $-u$. The velocity of A becomes $u-u=0$, and the point A is brought to rest. The changed velocity of B is now $V-u=v$, in the vectorial sense, and this is the velocity of B relative to A, which has become stationary.

The difference in the point of view is merely one of form. When the vectorial equation $u+v=V$ is applied to a point having the two component velocities u, v, this is equally consistent with supposing that the point in question has the velocity v relative to another point, which in turn has the velocity u.

64. All motion, if it is perceptible (and otherwise it has no meaning), implies a relative change of position between one object and another, which may be taken as fixed. On the latter can be based a fixed frame of reference, as already explained, but this frame is fixed only in the sense that it is not necessary to take any interest in the motion which it may possess. In the universe there is no fixed frame, and absolute motion is a phrase devoid of meaning. All motion, all velocity, all acceleration, is relative. In kinematics a fixed frame can be adopted according to convenience. In dynamics the case is more difficult, and the application of ordinary laws involves an arbitrary choice. It is then necessary to find a frame which, for the purposes of the particular problem, can be treated as stationary. In ordinary experiments the room in which they are performed may be so considered. In special cases and in large-scale experiments (such as the trajectories of shells) the earth's rotation needs to be taken into account, at least theoretically, while the earth's orbital motion can be ignored. For the motions of planets in

the solar system that system as a whole is taken to be at rest, though it has a measurable motion relative to any large group of stars. To this sequence there is no finality, except that which is imposed by the impossibility of tracing it indefinitely.

65. *Ex.* To calculate the resultant of velocities 10·7, 5·3, 9·8, making respectively with a fixed direction the angles 17°·2, 124°·7, 250°·1.

	log.	$u\cos\alpha$.		log.	$u\sin\alpha$.
10·7	1·0294		10·7	1·0294	
cos 17°·2	9·9801		sin 17·2	9·4709	
	1·0095	+10·22		0·5003	+3·16
5·3	0·7243		5·3	0·7243	
cos 124°·7	$9·7553_n$		sin 124°·7	9·9150	
	$0·4798_n$	−3·02		0·6393	+4·36
9·8	0·9912		9·8	0·9912	
cos 250°·1	$9·5320_n$		sin 250°·1	$9·9733_n$	
	$0·5232_n$	−3·34		$0·9645_n$	−9·22
$V\cos\beta$	0·5866	+3·86	$V\sin\beta$	$0·2304_n$	−1·70
$\cos\beta$	9·9615		$V\cos\beta$	0·5866	
	0·6251	$V=4·22$	$\tan\beta$	$9·6438_n$	$\beta=336°·2$

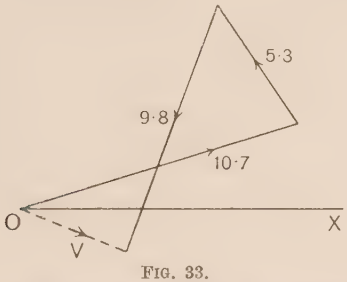

FIG. 33.

The resultant velocity is therefore 4·22, making the angle 336°·2 with the fixed direction OX. The result can be checked roughly by plotting the vectors as in the diagram.

66. *Ex.* A pedestrian, walking with the speed v, wishes to cross a road as far as possible in front of a car, approaching with the speed $V(>v)$. What direction should he take?

The conditions of the problem are not altered by assigning the opposite velocity V to the man and the car, whereby the latter is brought to rest. Relative to the car, the man now has one velocity $AO=V$ and another v, which, being given in magnitude and optional in direction, can be represented by a radius of a given circle. If OT be the radius in the chosen direction, the resultant relative velocity is

$$AO+OT=AT$$

in magnitude and direction. The car being now treated as at rest, the solution is clearly found by making the angle OAT as large as possible,

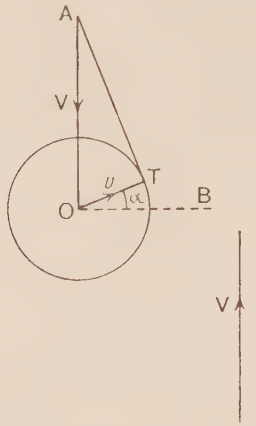

FIG. 34.

which means that AT is a tangent to the circle of radius v. Hence, if α be the angle between OT, the direction of walking, and OB, drawn perpendicular to the direction OA of the approaching car,

$$\sin \alpha = \sin BOT = \sin OAT = v/V,$$

and this gives the direction to be taken.

67. *Ex.* A boatman, whose speed of rowing is u, crosses a river of width b to a point at a distance a upstream and returns. Find the time of the double journey, if the rate of the stream is $V\,(<u)$.

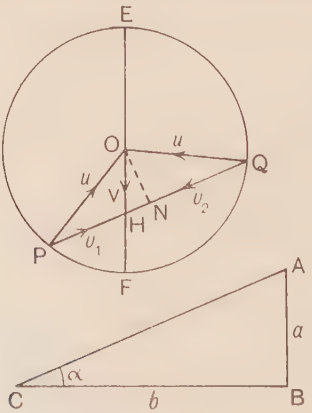

FIG. 35.

Relative to the water, the speed u is given, but the direction can be chosen. The corresponding velocity can therefore be represented by

the radius of a circle. The velocity of the stream V is given in magnitude and direction. The direction of the resultant is that of the actual course CA, where C is the starting point and A is the destination.

Hence the velocity vectors are related as in the figure. The stream velocity is $OH = V$. The chord PHQ is drawn parallel to the actual course CA. On the outward journey $PO = u$ combined with V gives the velocity $PH = v_1$, and on the return journey $QO = u$ combined with V gives the velocity $QH = v_2$. Let N be the foot of the perpendicular from O on PQ, or the middle point of PQ. Since ON is perpendicular to CA and OH is perpendicular to CB, $NOH = ACB = \alpha$. The time of the double journey is

$$T = \frac{CA}{v_1} + \frac{CA}{v_2} = \frac{CA}{PH \,.\, HQ} \,.\, PQ$$

$$= \frac{2CA}{EH \,.\, HF} \,.\, NQ = \frac{2CA}{u^2 - V^2} \sqrt{(u^2 - V^2 \cos^2 \alpha)}$$

$$= \frac{2}{u^2 - V^2} \sqrt{\{(a^2 + b^2) u^2 - b^2 V^2\}}.$$

This time is clearly shortest when $a = 0$, or when the crossing is made directly to the opposite point, in which case

$$T' = \frac{2b}{\sqrt{(u^2 - V^2)}} > \frac{2b}{u};$$

while for the same distance b up and down stream, the time is

$$T'' = \frac{b}{u - V} + \frac{b}{u + V} = \frac{2bu}{u^2 - V^2},$$

which is still greater. In fact, it is easily seen that

$$T' = T'' \,.\, \sqrt{\left(1 - \frac{V^2}{u^2}\right)}.$$

Like most problems this can be treated in more than one way, and it will be found an instructive exercise to devise another form of solution.

68. *Ex.* An aeroplane, with given speed V in air, flies round a closed circuit in time T when there is no wind. With a steady wind blowing with a velocity $kV\,(0 < k < 1)$, prove that the time round the circuit will be between $T(1 - k^2)^{-\frac{1}{2}}$ and $T(1 - k^2)^{-1}$.

Let the axis OX be taken in the direction of the wind, and let the tangent at P to the circuit make the angle ψ with OX. This is the angle which the velocity V' relative to the ground makes with kV. Hence

$$V^2 = V'^2 + k^2 V^2 - 2kV' \cos \psi$$

$$= (V' - kV \cos \psi)^2 + k^2 V^2 \sin^2 \psi,$$

so that

$$V' = V(1 - k^2 \sin^2 \psi)^{\frac{1}{2}} + kV \cos \psi.$$

A small arc δs of the circuit at P is described in the time $\delta s/V'$, and the whole circuit will be completed in the time

$$t = \Sigma \frac{\delta s}{V} \{(1 - k^2 \sin^2 \psi)^{\frac{1}{2}} + k \cos \psi\}^{-1}$$

$$= \frac{1}{V(1-k^2)} \Sigma \, \delta s \, \{(1 - k^2 \sin^2 \psi)^{\frac{1}{2}} - k \cos \psi\},$$

where the summation is taken round the whole circuit and the small arcs δs are reduced in length and multiplied in number indefinitely.

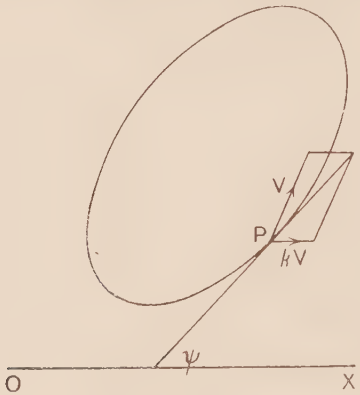

FIG. 36.

But $\delta s \,.\, \cos \psi = \delta x$ is the projection of the arc on OX, and $\Sigma \, \delta x = 0$, being the projection of a closed circuit. Hence

$$t = \frac{1}{V(1-k^2)} \Sigma \, \delta s \, (1 - k^2 \sin^2 \psi)^{\frac{1}{2}},$$

while, when $k = 0$,

$$T = \frac{1}{V} \Sigma \, \delta s.$$

Every element in the sum is positive, and therefore clearly

$$t < \frac{1}{V(1-k^2)} \Sigma \, \delta s < \frac{T}{1-k^2};$$

and, on the other hand,

$$t > \frac{1}{V(1-k^2)} \Sigma \, \delta s \, (1-k^2)^{\frac{1}{2}} > \frac{T}{(1-k^2)^{\frac{1}{2}}}.$$

Thus

$$T(1-k^2)^{-\frac{1}{2}} < t < T(1-k^2)^{-1},$$

which is the result required.

69. The simplest type of motion in a plane, which is not one of uniform velocity and is not restricted to a straight line, is that in which the point moves with uniform acceleration. Like velocity, acceleration is a directed quantity, and uniformity implies a constant

direction as well as a constant magnitude. The motion of an ordinary projectile *in vacuo* satisfies this condition, and the problem will be discussed in this form. The magnitude of the acceleration is therefore represented by g, and its direction is vertically downwards.

Let O be the point of projection, or the position of the point at $t=0$, and let $OT=u$ represent the initial velocity. Let OX be the

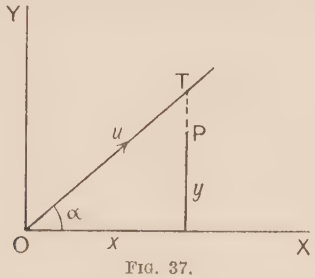

FIG. 37.

horizontal line at right angles to OY in the plane YOT. The angle $XOT=\alpha$ is the angle of projection. As the velocity and acceleration of the point have no component at right angles to the plane XOY, the motion of the point will be confined to this plane. The components of the displacement of the moving point P are its coordinates (x, y), of its velocity $(\dot{x}, \dot{y})$, and of its acceleration $(\ddot{x}, \ddot{y})$ at any time t. The fundamental conditions of the motion are represented by

$$\ddot{x}=0, \quad \ddot{y}=-g,$$

while initially $(t=0)$

$$x=y=0, \quad \dot{x}=u\cos\alpha, \quad \dot{y}=u\sin\alpha.$$

Hence at any time t,

$$\dot{x}=u\cos\alpha, \qquad \dot{y}=u\sin\alpha-gt,$$
$$x=u\cos\alpha\,.\,t, \quad y=u\sin\alpha\,.\,t-\tfrac{1}{2}gt^2,$$

the horizontal and vertical motions being defined by the equations independently. The relation which connects x and y for all values of t is

$$y=x\tan\alpha-\tfrac{1}{2}gx^2/u^2\cos^2\alpha,$$

and this defines the path or trajectory. The second term on the right of this equation represents the fall of the point P below the line OT.

70. It will now be assumed that α is a positive, acute angle, and several deductions can be easily made. The point P, starting from O at $t=0$ reaches OX once again when $y=0$ or

$$0=x(\tan\alpha-\tfrac{1}{2}gx/u^2\cos^2\alpha),$$
$$x=2\tan\alpha\,.\,u^2\cos^2\alpha/g=u^2\sin 2\alpha/g,$$

and this is the horizontal range. When u, the velocity of projection, is given, the range is evidently greatest when $\sin 2\alpha = 1$, or $\alpha = 45°$, so that the maximum range is u^2/g.

The time of flight, or time required to return to the horizontal plane, is

$$t = \frac{x}{u\cos\alpha} = \frac{u^2 \sin 2\alpha}{gu\cos\alpha} = 2u\sin\alpha/g.$$

The vertical velocity, $\dot{y} = u\sin\alpha - gt$, is at first positive and decreasing until it vanishes at $t = u\sin\alpha/g$, after which it remains negative or downwards. At the instant when it vanishes the velocity is entirely horizontal, and

$$x = u\cos\alpha \,.\, t = \tfrac{1}{2}u^2 \sin 2\alpha/g,$$
$$y = (u\sin\alpha - \tfrac{1}{2}gt)t = \tfrac{1}{2}u^2\sin^2\alpha/g.$$

This is the highest point reached in the trajectory, and it occurs midway in time and horizontal distance towards the end of the horizontal range.

71. This suggests a symmetry of which advantage may be taken by reckoning the time t' from this instant, so that

$$t = u\sin\alpha/g + t',$$

and

$$\dot{x} = u\cos\alpha, \quad \dot{y} = -gt',$$
$$x = \tfrac{1}{2}u^2\sin 2\alpha/g + u\cos\alpha \,.\, t',$$
$$y = (\tfrac{1}{2}u\sin\alpha - \tfrac{1}{2}gt')(u\sin\alpha/g + t')$$
$$= \tfrac{1}{2}u^2\sin^2\alpha/g - \tfrac{1}{2}gt'^2.$$

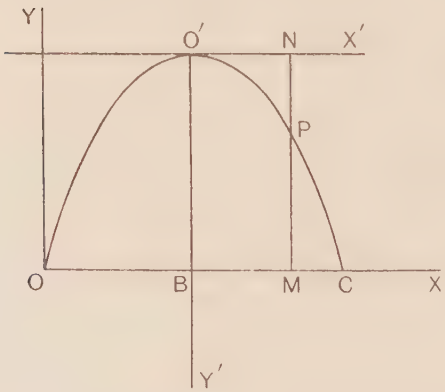

FIG. 38.

The relation connecting x, y now takes the form

$$(x - \tfrac{1}{2}u^2\sin 2\alpha/g)^2 = u^2\cos^2\alpha \,.\, t'^2$$
$$= -(y - \tfrac{1}{2}u^2\sin^2\alpha/g)2u^2\cos^2\alpha/g.$$

Draw the line $O'X'$ parallel to OX at the height $\frac{1}{2}u^2 \sin^2 \alpha/g$ above O, and the perpendicular line $O'Y'$ vertically downwards and cutting OX in B, where $OB = \frac{1}{2}u^2 \sin 2\alpha/g$. The position of a point P on the trajectory can be defined either by its original coordinates $OM = x$, $MP = y$, or relatively to the axes $O'X'$, $O'Y'$ by the coordinates $O'N = x'$, $NP = y'$; and clearly

$$x = OB + x' = x' + \tfrac{1}{2}u^2 \sin 2\alpha/g,$$
$$y + y' = MN = \tfrac{1}{2}u^2 \sin^2 \alpha/g.$$

The relation between x and y now becomes

$$x'^2 = y' \,.\, 2u^2 \cos^2 \alpha/g,$$

the form of which shows that the trajectory is a parabola, of which $O'Y'$ is the axis and $O'X'$ is the tangent at the vertex. In terms of t' the new coordinates are

$$x' = u \cos \alpha \,.\, t', \quad y' = \tfrac{1}{2}gt'^2,$$

while the corresponding components of velocity at the same point are

$$\dot{x}' = u \cos \alpha, \quad \dot{y}' = gt'.$$

All these results would be obtained immediately by supposing the moving point to start from O' at the time $t' = 0$ in the horizontal direction with the velocity $u \cos \alpha$.

72. Referred to the axes originally chosen, the position of the moving point is given by

$$x = u \cos \alpha \,.\, t, \quad y = u \sin \alpha \,.\, t - \tfrac{1}{2}gt^2,$$

and these equations can always be written down easily, because they express naturally the effect of the initial velocity together with the fall due to the vertical acceleration g.

If (x, y) define the position of a point through which the trajectory is to pass, u being given,

$$y = x \tan \alpha - \tfrac{1}{2}gx^2(1 + \tan^2 \alpha)/u^2$$

is the equation which must be satisfied by α. This quadratic in $\tan \alpha$, which may be written

$$\tan^2 \alpha - 2u^2 \tan \alpha/gx + 2u^2 y/gx^2 + 1 = 0,$$

has in general two roots, $\tan \alpha_1$ and $\tan \alpha_2$, where

$$\tan \alpha_1 + \tan \alpha_2 = 2u^2/gx,$$
$$\tan \alpha_1 \tan \alpha_2 = 2u^2 y/gx^2 + 1.$$

Hence

$$\tan (\alpha_1 + \alpha_2) = \frac{2u^2/gx}{-2u^2 y/gx^2} = -\frac{x}{y} = -\cot \beta,$$

where β is the elevation of the point (x, y) from O, the point of projection, so that $x=r\cos\beta$, $y=r\sin\beta$, r being the range. Thus

$$\cos(\alpha_1+\alpha_2-\beta)=0, \quad \alpha_1+\alpha_2=\beta+90^\circ,$$

and this is the simple relation connecting the two angles of projection by which one point may be reached from another with given initial velocity.

73. The result of eliminating α, instead of t, from

$$x=u\cos\alpha\,.\,t, \quad y=u\sin\alpha\,.\,t-\tfrac{1}{2}gt^2,$$

is

$$x^2+(y+\tfrac{1}{2}gt^2)^2=u^2t^2,$$

and this shows that if points start simultaneously from O in all directions in a vertical plane with the velocity u, they will be found after a time t on a circle of radius ut with its centre at $(0, -\frac{1}{2}gt^2)$. But the same equation, written as a quadratic in t^2,

$$\tfrac{1}{4}g^2t^4-(u^2-gy)t^2+x^2+y^2=0,$$

gives the two times, t_1, t_2, in which the point (x, y) can be reached from O with the same initial velocity u. These correspond with the two trajectories which are possible. The product of the roots of the quadratic is

$$t_1^2t_2^2=4(x^2+y^2)/g^2=4r^2/g^2,$$

so that $t_1t_2=2r/g$ depends on the range only, and is independent of the elevation β and the velocity of projection u.

74. The quadratics in $\tan\alpha$ and in t^2 found above for the two intersecting trajectories can be written in the form

$$\left(x\tan\alpha-\frac{u^2}{g}\right)^2=\frac{u^4}{g^2}-\frac{2u^2y}{g}-x^2,$$

$$\left\{\tfrac{1}{2}gt^2-\left(\frac{u^2}{g}-y\right)\right\}^2=\frac{u^4}{g^2}-\frac{2u^2y}{g}-x^2.$$

From either it follows that two trajectories are possible, provided

$$x^2\leq\frac{2u^2}{g}\left(\frac{u^2}{2g}-y\right),$$

the two coinciding when this is an equality. Otherwise a point starting from O with the velocity u cannot reach the point (x, y) at all.

Draw the horizontal line $O'X'$ at a height $OO'=\frac{1}{2}u^2/g$ above OX. Relatively to $O'X'$, $O'O$ the position of a point P (x, y) can be defined by $O'N=x$, $NP=y'=\frac{1}{2}u^2/g-y$. The boundary of accessible points just found is given by

$$x^2=y'\,.\,2u^2/g=4y'\,.\,OO',$$

which represents a parabola of which $O'O$ is the axis and O is the focus. The coordinates of all points within this parabola clearly satisfy the inequality $x^2 < y' \cdot 2u^2/g$ and are accessible by two distinct trajectories starting from O with the velocity u. The

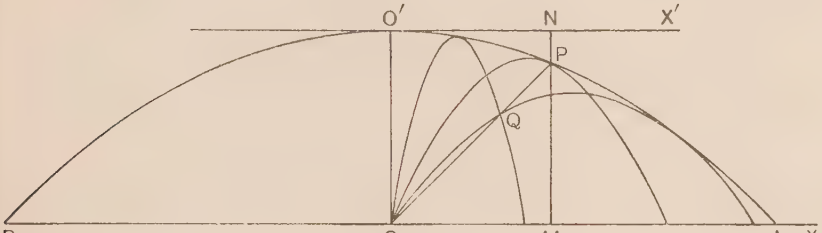

FIG. 39.

trajectories become coincident when the point is on the bounding parabola $AO'B$, and where they meet this boundary they touch it, since they cannot cross it. The parabolic boundary is thus the *envelope* of the trajectories, and it represents approximately the visible form assumed by a fountain.

75. Let P be a point on the limiting parabola, and let $OP = r$, $AOP = \beta$, so that for this point $x = r\cos\beta$, $y = r\sin\beta$. It is evident that r is the greatest range corresponding to the elevation β. Then, from the relation satisfied by x, y,

$$r^2\cos^2\beta + \frac{2u^2}{g} \cdot r\sin\beta - \frac{u^4}{g^2} = 0,$$

$$\left\{r(1-\sin\beta) + \frac{u^2}{g}\right\}\left\{r(1+\sin\beta) - \frac{u^2}{g}\right\} = 0,$$

$$r = \frac{u^2}{g(1+\sin\beta)}.$$

Let α be the angle of projection required to obtain this maximum range, and let α_1, α_2 be the angles of projection for the two trajectories which pass through the point Q on OP. In the case of P two trajectories coincide, with equal values of α. Hence

$$\alpha_1 + \alpha_2 = \beta + 90° = 2\alpha,$$

from which it follows that $\alpha = 45° + \frac{1}{2}\beta$ gives the maximum range in the direction β, and that at O the corresponding trajectory bisects the angle between the two trajectories which cross one another again at any point Q on OP. Also, since the times in these two trajectories are connected by the relation $t_1 t_2 = 2OQ/g$, when

the two pass through P and coincide, the time over the arc OP is given by

$$t^2=\frac{2r}{g}=\frac{2u^2}{g^2(1+\sin\beta)},$$

whence the time over the maximum range can also be written $t=u/g\sin\alpha$.

76. One important remark may be made at this point. The problem of the simple parabolic trajectory, in the form adopted here, involves two physical quantities only, the acceleration g and the velocity u. The dimensions of g are LT^{-2} and those of u are LT^{-1}, where L is a length and T a time. In order to produce a length, g and u must be combined in such a way as to eliminate T, and the only possible combination is u^2/g. Similarly, the only possible combination which will produce a time is u/g. It will be noticed that these combinations, multiplied by a numerical factor without physical dimensions, do in fact occur in the results.

More generally, every term of an equation has the physical dimensions which result from its factors, and these dimensions must always be the same. This fact, generally visible on inspection, affords a useful check on the accuracy of an equation and must always be borne in mind. For this reason the premature insertion of numerical values in an equation is to be deprecated, because the check is thereby invalidated. Fuller discussion will be found in Chapter V.

Ex. The velocity of projection being given, find the trajectory of slowest descent to a downward slope.

If u is the velocity, α the angle of projection, r the range, β the downward slope and t the time of flight,

$$x=r\cos\beta=u\cos\alpha\,.\,t,$$

$$y=-r\sin\beta=u\sin\alpha\,.\,t-\tfrac{1}{2}gt^2,$$

whence
$$u\cos\alpha\sin\beta\,.\,t+(u\sin\alpha\,.\,t-\tfrac{1}{2}gt^2)\cos\beta=0,$$

$$t=\frac{2u}{g}\,.\,\frac{\sin(\alpha+\beta)}{\cos\beta}.$$

This is clearly greatest when $\sin(\alpha+\beta)=1$, $\alpha=90^\circ-\beta$, and

$$t=2u/g\cos\beta.$$

The corresponding range is $r=u\cos\alpha\sec\beta\,.\,t=2u^2\sin\beta/g\cos^2\beta$. This trajectory starts at right angles to the given slope, while the other trajectory which gives the same range starts horizontally, the corresponding time of flight being $2r/gt=2u\tan\beta/g$.

77. Another simple type of motion is presented by a point describing a circle uniformly. Let a be the radius of the circle,

centre O, and let n be the uniform angular velocity. If the point P crosses the line OX when $t=0$, the angle $XOP=nt$, and the co-ordinates of P at the time t are

$$x=a\cos nt, \quad y=a\sin nt.$$

The constant speed of P has been seen to be an, and its direction is along the tangent to the circle, or at right angles to the radius OP.

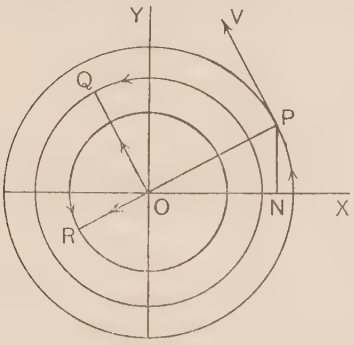

FIG. 40.

If then PV represents the velocity, PV makes with OX the angle $90°+nt$, and the components of PV are

$$\dot{x}=-an\sin nt=-ny,$$
$$\dot{y}=\quad an\cos nt=\quad nx.$$

Hence if OQ is drawn equal and parallel to PV, Q describes a concentric circle of radius an, and at any time is 90° in advance of P. The components of acceleration of P follow at once, for

$$\ddot{x}=-\frac{d}{dt}(ny)=-n\dot{y}=-n^2x,$$

$$\ddot{y}=\quad\frac{d}{dt}(nx)=\quad n\dot{x}=-n^2y.$$

Hence if OR be drawn to represent the acceleration of P in magnitude and direction, R also describes a concentric circle of radius an^2, and at any time is diametrically opposite to, or 180° in advance of, P. It will be noticed that the components of the velocity and acceleration of P, $\dot{x}$ and $\ddot{x}$, agree with the results previously obtained by direct differentiation for the simple harmonic motion, which is the motion of N, the projection of P on OX. Also the acceleration OR bears the same relation to the velocity OQ as OQ bears to the displacement OP, and might have been deduced by this principle.

78. The most important result has now been obtained that the acceleration of a point P, describing a circle uniformly, is in the direction OR or PO, that is, directly towards the centre of the circle. Also, if $v=an$ is the uniform speed of P, the magnitude of this acceleration, which is naturally constant, is $OR=an^2=v^2/a$. It should be obvious that the point P possesses acceleration, for without it the velocity of P would be uniform, and this implies motion in a straight line. Further, since the acceleration, of dimensions LT^{-2}, must depend only on v, which has the dimensions LT^{-1}, and on a, which has the dimensions L, the magnitude of the acceleration can only be of the form v^2/a multiplied by a numerical factor without dimensions. It is now seen that this factor is 1, when it is assumed that all the quantities are expressed in consistent units.

The circle described by the point Q is the locus of the extremity of the velocity vector PV transferred parallel to itself to a fixed point O. Such a locus is called the *hodograph* of the motion of P. It has two arbitrary elements, one of scale and one of position. Its scale is that on which its radii represent the parallel velocities, and this is clearly arbitrary. The fixed point O, which is taken as the common origin of the transferred vectors, is also arbitrary; it has been merely convenient in the above case to make it coincide with the centre of the circle described by the moving point.

The hodograph is not a mere curve, but a curve described with a characteristic motion. Hence it is possible to repeat the process, and to derive the locus of R, which is the hodograph of the motion of Q, or the hodograph of the hodograph of the motion of P. The radii of this curve represent the accclerations of P in the same way as the radii of the first hodograph represent the velocities of P, though this property will require further examination in the general case. The scale of the second hodograph is not arbitrary in the same sense as that of the first. Thus, in the particular case of uniform circular motion, the radii a and an can be represented independently, but the radius an^2 is a third proportional. This corresponds with the fact that length, velocity and acceleration are expressed in consistent units.

Ex. In the case of simple parabolic motion, the initial point of the hodograph is A, the extremity of the velocity vector u.

The acceleration being uniform, a vertical velocity proportional to the time is afterwards added, so that the hodograph is the vertical

line AB, uniformly described by the point Q. The uniform velocity of Q, represented by OR, is the acceleration in the original motion, and

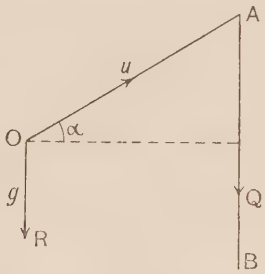

FIG. 41.

since OR is constant, the hodograph of the hodograph is here the fixed point R.

79. The motion of a point P along a curve in a plane will now be considered in more general terms. Let the point move from the position P at the time t to the neighbouring point P' at the time

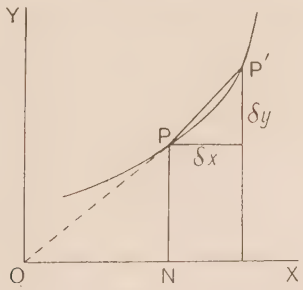

FIG. 42.

$t+\delta t$, differing little from t. If OX, OY are two fixed rectangular axes in the plane, the coordinates (x, y) of P may be regarded as the components of the displacement OP of P from O. Similarly, the displacement PP' may be represented by its components $(\delta x, \delta y)$. The rates at which these are increasing at P must be found by proceeding to limits, and

$$\dot{x} = \text{Lim.}\, \frac{\delta x}{\delta t} = \frac{dx}{dt},$$

$$\dot{y} = \text{Lim.}\, \frac{\delta y}{\delta t} = \frac{dy}{dt},$$

are the components of the velocity at P. Otherwise, if N is the foot of the ordinate from P, the velocity of P (relative to O) is the

resultant of the velocity $\dot{x}$ of N relative to O, and the velocity $\dot{y}$ of P relative to N. Now

$$\frac{\dot{y}}{\dot{x}} = \text{Lim.} \left(\frac{\delta y}{\delta t} \div \frac{\delta x}{\delta t} \right) = \text{Lim.} \frac{\delta y}{\delta x} = \frac{dy}{dx}.$$

But dy/dx is the tangent of the angle which the tangent to the curve at P makes with OX (cf. § 40). This shows more clearly, what before has been assumed, that the velocity, of which the components are $(\dot{x}, \dot{y})$, is in the direction of the tangent to the curve.

More directly, the displacement in the interval δt is PP', of which the direction tends to that of the tangent at P, the more closely P' approaches P. Let δc be the chord PP' and δs the arc PP'; in the same circumstances $\delta c/\delta s \to 1$. The magnitude of the velocity at P is

$$\text{Lim.} \frac{\delta c}{\delta t} = \text{Lim.} \left(\frac{\delta c}{\delta s} \cdot \frac{\delta s}{\delta t} \right) = \text{Lim.} \frac{\delta s}{\delta t} = \frac{ds}{dt} = \dot{s},$$

where $\dot{s}$ has been previously defined (§ 58) as the speed in traversing the curve as a mere line, regardless of its curvature. This speed is now seen to be the magnitude of the velocity of P in the plane, with the two components $\dot{x}$, $\dot{y}$. Hence

$$\dot{s}^2 = \dot{x}^2 + \dot{y}^2,$$

or

$$\left(\frac{ds}{dt} \right)^2 = \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2.$$

80. Because linear displacements, velocities and accelerations are vectors, their expression by means of coordinates or components parallel to fixed axes are particularly simple and immediate. But other forms are possible and often more convenient.

When the position of the moving point P is defined by the distance $OP = r$ from a fixed point O and the angle $XOP = \theta$ which OP makes with a fixed straight line OX, the polar coordinates (r, θ) vary with t as P describes a curve. From the rates $(\dot{r}, \dot{\theta})$ at which these vary, the velocity of P at the time t can be determined. The problem has already been stated (§ 59), and it can now be solved.

Suppose that OP at the time t coincides with another line OP_0, r in length, which is rotating uniformly with the angular velocity $\dot{\theta}$. The point P_0 is then describing a circle of radius r uniformly, and its velocity is $r\dot{\theta}$ in the direction of the tangent, or at right angles to OP_0. But the point P carried on OP, which at the instant has the same angular velocity as OP_0, is at the same time moving along OP with the velocity $\dot{r}$. Hence the velocity of P (relative to O),

being the resultant of the velocity of P relative to P_0 and the velocity of P_0 relative to O, is compounded of the radial velocity $\dot{r}$ along OP

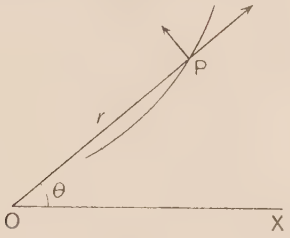

FIG. 43.

together with the transversal velocity $r\dot{\theta}$ in the direction at right angles. Since the resultant is $\dot{s}$ as before,

$$\dot{s}^2=\dot{r}^2+r^2\dot{\theta}^2,$$

or

$$\left(\frac{ds}{dt}\right)^2=\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2.$$

81. The components of the acceleration of P are $(\ddot{x}, \ddot{y})$, where

$$\ddot{x}=\frac{d^2x}{dt^2}, \quad \ddot{y}=\frac{d^2y}{dt^2},$$

for these represent respectively the acceleration of N, the foot of the ordinate through P, relative to O and the acceleration of P relative to N. But, as the case of uniform circular motion clearly shows, the direction of the resultant acceleration is not that of the tangent to the path of P; nor is its magnitude represented by $\ddot{s}$.

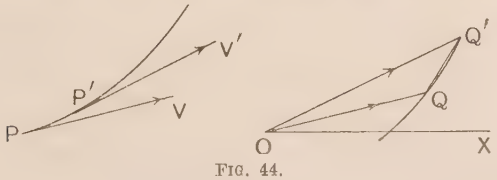

FIG. 44.

Let the moving point pass from the position P at the time t to P' at the time $t+\delta t$, and let the tangents PV, $P'V'$ be the corresponding tangents to the path, drawn to represent the velocities. The arc PP' will be denoted by δs and the inclination of $P'V'$ to PV by $\delta\psi$; the angle ψ may be reckoned from any fixed direction OX. The velocity vectors PV, $P'V'$ being transferred to the parallel positions OQ, OQ', the locus of Q will be the hodograph of the motion. For points on this curve, $OQ=PV=v$, $XOQ=\psi$ are polar coordinates in the ordinary sense.

In the interval δt, OQ changes to OQ', and in the vectorial sense $OQ' - OQ = QQ'$ represents the change of velocity in magnitude and direction. In the limit, when δt is reduced indefinitely, QQ' becomes the tangent to the hodograph at Q. Also if $\delta\sigma$ is the arc QQ', which is ultimately equal to the chord,

$$\dot{\sigma} = \text{Lim.}\frac{\delta\sigma}{\delta t} = \frac{d\sigma}{dt}$$

represents the rate of change of σ, and equally the velocity of Q in the hodograph or the rate of change of the velocity of P in its path, that is, the acceleration of P at the time t.

But since (v, ψ) are the polar coordinates of Q in the hodograph, the velocity of Q consists of a radial component $\dot{v}$ and a transversal component $v\dot{\psi}$. The first, $\dot{v}$, being along OQ or parallel to PV, is therefore the tangential component of the acceleration of P. Similarly, the second, being at right angles to OQ or PV is the inward normal component of the acceleration of P. Now $v = \dot{s}$ is the velocity of P, and

$$\dot{v} = \frac{d\dot{s}}{dt} = \ddot{s} = v\frac{dv}{ds}$$

are therefore equivalent forms for the tangential acceleration. Incidentally this assigns a precise meaning to $\ddot{s}$, previously introduced (§ 58) to express the acceleration of P in the curve without regard to its motion in space. Further, the normal component is

$$v\dot{\psi} = v\,\text{Lim.}\frac{\delta\psi}{\delta t} = v\,\text{Lim.}\left(\frac{\delta s}{\delta t} \div \frac{\delta s}{\delta\psi}\right)$$
$$= v^2/\rho,$$

where ρ is the radius of curvature of the path at P (cf. § 21).

82.* The components along the tangent and normal to the path naturally express the acceleration of the moving point in terms most directly related to the geometrical character of the curve described. Another form is obtained in terms of the polar coordinates, by which the position of the point may be defined, and their variations with the time. As in the case of the velocity already considered, the components are radial and transversal.

At the time t the position of the point P is defined by $OP = r$, $XOP = \theta$. The motion of P along OP is defined by the variation of r, while the line OP rotates in a way defined by the variation of θ. Let this line carry a fixed point at the constant distance r from O, and let it come to P_0 when P reaches P' at the time $t + \delta t$. The

* The rest of this chapter may be omitted on first reading.

angle POP' may be denoted by $\delta\theta$ and the distance P_0P' by δr. The point P_0 moves in a circle of radius r, though not uniformly, and in this motion the arc from a fixed point P is $s=r\theta$, $\dot{s}=r\dot{\theta}$, $\ddot{s}=r\ddot{\theta}$; the component of acceleration along the radius PO is there-

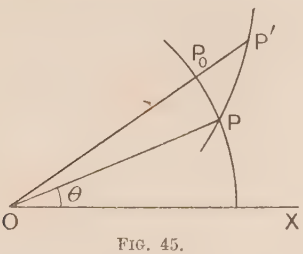

FIG. 45.

fore $\dot{s}^2/r=r\dot{\theta}^2$. But the acceleration of P along OP, treated as line not rotating, is $\ddot{r}$, and this is the radial acceleration of P relative to P_0 at the moment of coincidence. Hence the radial component of the acceleration of P relative to the fixed point O is

$$R=\ddot{r}-r\dot{\theta}^2.$$

The transversal component of the acceleration needs more careful treatment. At the instant t, when P and P_0 coincide, the velocity of P relative to P_0 is $\dot{r}$ in the radial direction OP. At P' the velocity relative to P_0 has a component at right angles to OP for two reasons. The relative velocity $\dot{r}$, but slightly changed, along OP' has a component $\dot{r}\sin(\delta\theta)=\dot{r}\,.\,\delta\theta$. Also $P_0P'=\delta r$ is rotating with angular velocity $\dot{\theta}$, and thus P' has a relative velocity $\delta r\,.\,\dot{\theta}$ at right angles to P_0P'. Hence the rate at which P' gains a velocity relative to P_0 in the direction transversal to OP is

$$\text{Lim.}\left(\dot{r}\frac{\delta\theta}{\delta t}+\dot{\theta}\frac{\delta r}{\delta t}\right)=2\dot{r}\dot{\theta},$$

which is therefore the transversal component of the acceleration of P relative to P_0. But the corresponding component of the acceleration of P_0 is $\ddot{s}=r\ddot{\theta}$. Hence the transversal component of the acceleration of P (relative to O) is

$$T=r\ddot{\theta}+2\dot{r}\dot{\theta}.$$

83. Another type of velocity, *areal velocity*, has been referred to in the previous chapter. This is the rate of increase of the area contained between the curve traced and two radii, one of which passes through the moving point; or rather, double this rate.

If the moving point passes from P at the time t to P' at the time $t+\delta t$, and in the interval δt the radius describes the small angle

$POP' = \delta\theta$, the area $POP' = \frac{1}{2}r^2 . \delta\theta$, where $OP < r < OP'$. Hence the double rate or areal velocity is

$$\text{Lim.}\, r^2\frac{\delta\theta}{\delta t} = r^2\frac{d\theta}{dt} = r^2\dot{\theta}.$$

Let PV represent the whole velocity $s = v$ of the moving point, and let $p = OD$ be the perpendicular from O on PV. The radial component of the velocity is $PR = \dot{r}$, the transversal component is

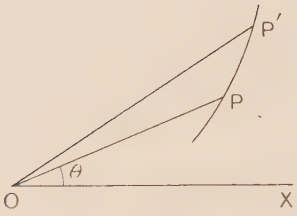

FIG. 46.

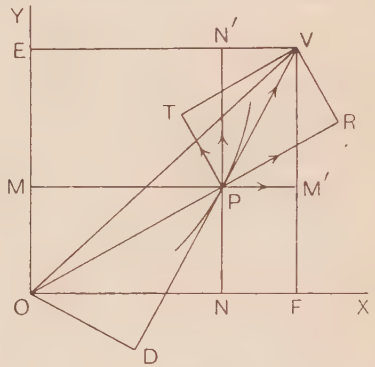

FIG. 47.

$PT = r\dot{\theta}$, and the components parallel to the axes are $PM' = \dot{x}$, $PN' = \dot{y}$: $EN'V$, MPM' being drawn parallel to OX, and $VM'F$, $N'PN$ parallel to OY.

The angles OPD, PVT, being the complements of VPT, are equal, and the triangles OPD, PVT are similar. Hence

$$p/r = OD/OP = PT/PV = r\dot{\theta}/v,$$

and the areal velocity is $r^2\dot{\theta} = pv$, or the product of the velocity into the perpendicular from O on the tangent to the path of P. This product $OD . PV$ is represented in the figure by twice the area of the triangle OPV. Now

$$\begin{aligned} 2OPV &= 2OFV - 2OPN - 2PVM' - 2PNFM' \\ &= OFVE - ONPM - PM'VN' - 2PNFM' \\ &= MPN'E - PNFM' \\ &= x\dot{y} - y\dot{x}. \end{aligned}$$

This, then, is another expression for the areal velocity, so that

$$r^2\dot{\theta} = pv = x\dot{y} - y\dot{x}$$

are all forms in which this velocity can be expressed.

84. It will now be shown that the areal velocity is related closely to the transversal component of the acceleration. Let two tangents

to the path meet in T, and let TV, TV' represent the velocities along them, so that in the vectorial sense $TV' - TV = VV'$ represents the change of velocity. Let N, N' be the feet of the perpendiculars from V, V' on the line drawn through T at right angles to OT. The areal velocities about O, if E, E' are the feet of the perpendiculars from O on TV, TV', are represented by

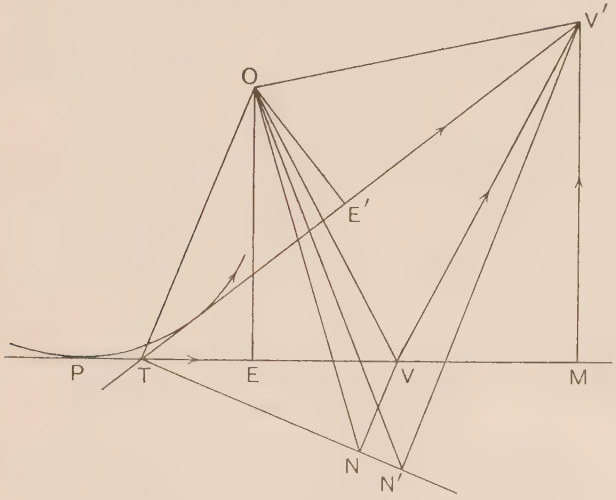

FIG. 48.

$$OE \,.\, TV = 2OTV = 2OTN$$

and $OE' \,.\, TV' = 2OTV' = 2OTN'$. The change in the areal velocity is therefore $2(OTN' - OTN) = 2ONN' = OT \,.\, NN'$, where NN' is the component of VV' at right angles to OT.

This result is true generally, in the sense that it has not been assumed that the tangents to the path are consecutive. It may be written:

$$\delta(r^2\dot{\theta}) = OT \,.\, \delta u,$$

where δu is the component NN' of VV', the change in the linear velocity. Hence if δt is the corresponding interval of time,

$$\text{Lim.}\, \frac{\delta(r^2\dot{\theta})}{\delta t} = \text{Lim.}\, OT \,.\, \frac{\delta u}{\delta t}.$$

Now in the limit, when the tangent TV' is made to approach TV, OT becomes $OP = r$, and the rate of change of u is the transversal component T of the acceleration. Hence

$$\frac{d}{dt}(r^2\dot{\theta}) = rT,$$

or again, with the earlier result,

$$T=r\ddot{\theta}+2\dot{r}\dot{\theta}=\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}).$$

The two forms are easily seen to be equivalent.

When, in particular, the areal velocity is constant, $T=0$. In this case the triangles OTV, OTV' are equal in area, and therefore the change in the velocity, VV', is parallel to OT. Hence the acceleration, which is the rate of this change, is along OP, or purely radial. In these circumstances the path of P is called a central orbit.

85. In the same figure let M be the foot of the perpendicular from V' on TV. Then, if $TV=v$, $TV'=v'$, $OE=p$, $OE'=p'$,

$$\begin{aligned} MV' &= TV'\sin MTV' \\ &= v'\sin(OTE-OTE') \\ &= \frac{v'}{r^2}\{p\sqrt{(r^2-p'^2)}-p'\sqrt{(r^2-p^2)}\} \\ &= \frac{v'}{r^2}\cdot\frac{p^2(r^2-p'^2)-p'^2(r^2-p^2)}{p\sqrt{(r^2-p'^2)}+p'\sqrt{(r^2-p^2)}} \\ &= \frac{v'(p^2-p'^2)}{p\sqrt{(r^2-p'^2)}+p'\sqrt{(r^2-p^2)}}. \end{aligned}$$

This is the component of the change of velocity VV' at right angles to TV, or parallel to the normal at P. Hence the component of the acceleration in this direction is

$$\begin{aligned} \text{Lim.}\frac{MV'}{\delta t} &= \text{Lim.}\left\{\frac{v'(p+p')}{p\sqrt{(r^2-p'^2)}+p'\sqrt{(r^2-p^2)}}\cdot\frac{p-p'}{\delta t}\right\} \\ &= \frac{v}{\sqrt{(r^2-p^2)}}\cdot\dot{p}. \end{aligned}$$

But from the previous figure 47 (p. 82)

$$\sqrt{(r^2-p^2)}=DP=OP\,.\,PR/PV=r\dot{r}/v,$$

while in the case represented here both the rates $\dot{r}$ and $\dot{p}$ are negative. Hence the inward normal acceleration at P is

$$v^2/\rho=v^2\dot{p}/r\dot{r}.$$

86. As in this last case, the expression of the normal acceleration under different forms leads to corresponding expressions for the radius of curvature. Here, for example,

$$\rho=\frac{r\dot{r}}{\dot{p}}.$$

Now it is to be observed that this result, although obtained by kinematical reasoning, is essentially geometrical in its character.

Its form would be the same if p and r were expressed in terms of any common parameter other than the time. For, as the time is irrelevant to the geometrical conditions, the varying parameter may itself be taken as the measure of the time and identified with it. Further, one of the two related quantities, as p, may be taken as the parameter, so that $\dot{p}=1$ and

$$\rho=r\frac{dr}{dp}.$$

The reverse process can also be illustrated. One expression found for the radius of curvature is

$$\rho=\frac{ds}{d\psi},$$

the limiting ratio of the arc to the angle through which the tangent turns. But this may be written

$$\rho=\text{Lim.}\left(\frac{\delta s}{\delta t}\div\frac{\delta\psi}{\delta t}\right)=\frac{\dot{s}}{\dot{\psi}},$$

or the ratio of the rate of increase of the arc to the corresponding rate of rotation of the tangent.

87. Parallel to rectangular axes the components of the velocity of a point are $(\dot{x}, \dot{y})$. If $\dot{s}=v$ is the velocity, the axes OX, OY make with the tangent angles whose cosines are $\dot{x}/v$, $\dot{y}/v$ and with the normal angles whose cosines are $-\dot{y}/v$, $\dot{x}/v$. The corresponding components of acceleration are $(\ddot{x}, \ddot{y})$. Hence the normal acceleration is

$$\frac{v^2}{\rho}=-\frac{\dot{y}}{v}\cdot\ddot{x}+\frac{\dot{x}}{v}\cdot\ddot{y},$$

and therefore

$$\rho=\frac{v^3}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}=\frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}.$$

In this formula the rates need not be taken with respect to the time, but with respect to any parameter in terms of which x and y are expressed. In particular, when this parameter is identified with x, so that $\dot{x}=1$, $\ddot{x}=0$,

$$\dot{y}=\frac{dy}{dx},\quad \ddot{y}=\frac{d^2y}{dx^2},\quad \rho=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

In polar coordinates, the radial component $\dot{r}$ and the transversal component $r\dot{\theta}$ of the velocity v make with the tangent angles whose cosines are $\dot{r}/v$, $r\dot{\theta}/v$, and with the normal angles whose cosines are

$r\dot{\theta}/v$ and $-\dot{r}/v$ respectively. The corresponding components of the acceleration are also known, and thus the normal acceleration is

$$\frac{v^2}{\rho}=\frac{r\dot{\theta}}{v}(\ddot{r}-r\dot{\theta}^2)-\frac{\dot{r}}{v}(r\ddot{\theta}+2\dot{r}\dot{\theta}),$$

whence, since $v^2=\dot{r}^2+r^2\dot{\theta}^2$,

$$\rho=\frac{(\dot{r}^2+r^2\dot{\theta}^2)^{\frac{3}{2}}}{r\ddot{r}\dot{\theta}-\dot{r}(2\dot{r}\dot{\theta}+r\ddot{\theta})-r^2\dot{\theta}^3}.$$

In this formula any parameter may be substituted for the time, and in particular θ may be taken as the parameter, in which case $\dot{\theta}=1$, $\ddot{\theta}=0$ and

$$\dot{r}=\frac{dr}{d\theta},\quad \ddot{r}=\frac{d^2r}{d\theta^2},\quad \rho=\frac{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r\frac{d^2r}{d\theta^2}-2\left(\frac{dr}{d\theta}\right)^2-r^2}.$$

Thus the most important formulae for the radius of curvature have been deduced from kinematical considerations.

88. In the case of a central orbit the areal velocity, $r^2\dot{\theta}=h$, is constant, the transversal acceleration $T=0$, and the radial component R to or from a fixed centre O represents the whole acceleration of the moving point P. In the hodograph of the motion let σ be the arc measured from a fixed point on the curve to the point Q which corresponds with P. The velocity of Q in the hodograph is therefore $\dot{\sigma}$, equal in magnitude to the acceleration R. This velocity, which is tangential to the hodograph at Q, is also parallel to R, or the radius vector OP, which is rotating at the rate $\dot{\theta}$. Hence the radius of curvature of the hodograph at Q is

$$\rho=\frac{\dot{\sigma}}{\dot{\theta}}=\frac{r^2R}{h}.$$

The most interesting case is that in which $R=\mu/r^2$, for this represents the conditions in an undisturbed gravitational orbit. Here $\rho=\mu/h$ is constant, and the hodograph is therefore a circle. This beautiful result is true, whether the central orbit is a planetary ellipse, a parabola or an hyperbola; the last may be described with acceleration towards or away from the fixed centre.

Since the areal velocity, in the form $pv=h$, is constant in the central orbit, the geometrical relation of the hodograph to the orbit is very simple. Let $ON=p$ be the perpendicular from the fixed centre O on a tangent NT to the orbit at P. On ON take a point B such that $OB\,.\,ON=h$ or $OB=h/p=v$. To every tangent to the

orbit corresponds a point B constructed in this way, and the locus of B is a derived curve called the *polar reciprocal* of the orbital curve. If the radius vector $OB=v$ be turned through a right angle into the

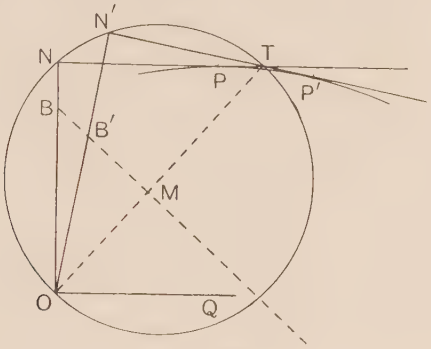

FIG. 49.

position OQ, $OQ=v$ is parallel to the tangent NT or to the velocity in the orbit at P. Hence the hodograph of the motion is the locus of Q, and it is the polar reciprocal of the orbit turned through a right angle.

It should be observed that this relation between the central orbit and its hodograph is a reciprocal one, or the orbit is also the polar reciprocal of the hodograph (with corresponding rotation). Let ON' be the perpendicular from O on a second tangent $N'T$ to the orbit, and let BM, the perpendicular from B to OT, meet ON' in B'. The angles at M, N, N' being right angles, the points $MTNB$ and the points $MTN'B'$ are concyclic, and

$$h=OB\,.\,ON=OM\,.\,OT=OB'\,.\,ON'.$$

Hence B' is also a point on the polar reciprocal of the orbit. If the tangent $N'T$ now tends to coincide with NT, B' tends to coincide with B and BB' becomes the tangent to the polar reciprocal of the orbit. At the same time T tends to coincide with P, and being a point on the perpendicular to the tangent BB' such that $OM\,.\,OT=h$, the locus of T, which in the limit is the orbit itself, is the polar reciprocal of the locus of B, which is the hodograph rotated through a right angle.

When the acceleration is radial and of the form μ/r^2, the hodograph is a circle and the orbit is the polar reciprocal of a circle. The reader who is familiar with this branch of geometry will need no further proof that the orbit is a conic section, of which the fixed centre is a focus.

EXAMPLES.

1. Calculate in magnitude and direction the resultant of the four accelerations 23·9 ft./sec.^{2} towards E.N.E., the same towards N.N.W., 33·8 ft./sec.^{2} towards W.N.W. and the same towards S.S.W.

2. A particle has an acceleration in a fixed direction: if ABC be the triangle formed by three tangents to its path, and u, v, w be the velocities at the points of contact of these tangents, show that

$$\frac{BC}{u}+\frac{CA}{v}+\frac{AB}{w}=0.$$

Is any convention necessary as to sign ?

3. A liner steaming V knots on a straight course passes a port without stopping, and a tender steaming $v(<V)$ knots is sent to intercept it. Show that the tender, in order to leave the port at the latest possible moment, must set its course at the angle $\cos^{-1}v/V$ to that of the liner, and that it must start when the direction of the liner is at right angles to this course.

4. An aeroplane which can complete a course in the form of an equilateral triangle in time $3T$ at a speed V when there is no wind, flies over the same course with a wind at velocity kV blowing directly across one side. Prove that the time will be

$$T(1-k^2)^{-1}\{(1-k^2)^{\frac{1}{2}}+(4-k^2)^{\frac{1}{2}}\},$$

and show that this is consistent with the general result of § 68.

5. If T is the time of flight in a parabolic trajectory over a horizontal range, prove that the height h at the time t is

$$h=\tfrac{1}{2}gt(T-t),$$

and that the height of the vertex in feet is (nearly) four times the square of the whole time of flight in seconds.

6. If α is the angle of projection from a point on an inclined plane, the inclination being β, and θ is the direction of flight when the particle reaches the plane again, prove that

$$\tan\theta=2\tan\beta-\tan\alpha.$$

Explain from dimensions why this relation between θ, α and β is independent of the initial velocity.

7. If the point of projection is at the vertical height h above an inclined plane, and u is the initial velocity, show from dimensions that the relation between α, the angle of projection, β, the inclination of the plane, and θ, the direction of flight when the particle meets the plane, must involve g, h and u only in the combination gh/u^2.

Find the relation in the form

$$(\tan\theta-\tan\alpha)(\tan\theta+\tan\alpha-2\tan\beta)=2gh/u^2\cos^2\alpha.$$

8. If two particles be projected from the same point at the same instant in the same vertical plane, with velocities v_1 and v_2 in directions making angles α_1 and α_2 with the horizon, prove that the interval of time between their transits through the other point which is common to their paths is

$$\frac{2v_1v_2}{g}\cdot\frac{\sin(\alpha_1-\alpha_2)}{v_1\cos\alpha_1+v_2\cos\alpha_2}.$$

9. When two trajectories with the same velocity of projection from A meet in a second point B, show that the velocity of arrival at B is the same by either path. Prove also that the bisector of the angle between the two paths at B is at right angles to the bisector of the corresponding angle at A.

10. If R is the greatest horizontal range possible without any restriction, show that $2\{2h(R-2h)\}^{\frac{1}{2}}$ is the greatest range attainable when the vertical height must not exceed h.

A cricketer can throw the ball 100 yds. What is the greatest distance he can throw it if the ball is not to rise more than 15 ft. in the air? Show that the angle of projection must be $\cot^{-1}3$.

11. In order to hit the highest point of a vertical wall with a given velocity of projection from a given point A on the ground in front of it, show that it is necessary to aim at a point B of the wall at a height above A equal to the greatest horizontal range, and that the point C reached on the wall will be such that $CA=CB$.

12. Trajectories start from a point O in all directions with the same initial velocity. Prove that the points at which the direction of flight is at right angles to the original direction of projection lie on the envelope to the trajectories.

13. Trajectories start from a point O in all directions with the same velocity in one vertical plane. If (x, y) are the coordinates of the vertex of any of these trajectories, the axes being OX horizontal and OY vertical, prove that

$$x^2+4y^2=2Ry,$$

where R is the greatest horizontal range.

14. Projectiles start from one point with initial velocities in one vertical plane such that the product of the rectangular components in the horizontal direction and in some other assigned direction is given. Prove that the trajectories will all pass through a second common point.

15. Prove that the radius of curvature of a parabolic trajectory at any point is $v^2/g\cos^3\theta$, where v is the constant horizontal velocity and θ is the inclination to the horizon at which the projectile is moving.

16. The speed of the seaplane which won the Schneider Cup in 1927 was such that a turn on a radius of one mile meant an acceleration equal to that due to gravity (g 32·2 ft./sec.^{2}). Deduce the speed.

17. A circular hoop is set spinning in a vertical plane and then allowed to fall freely. Show that with a suitable rate of spin the

lowest point of the hoop has no acceleration, and that in this case the acceleration of every other point on the hoop is directed towards the lowest point.

With $g = 981$ cm./sec.^{2} find the diameter of the hoop when one revolution per second gives these conditions.

18. A point moves in a plane with acceleration to a fixed point O according to the law μr^p, μ being constant. When it is projected so as to describe a circular orbit of radius a about O, find its period, its speed and its areal velocity. For what values of p are these three quantities respectively constant for orbits of any size ?

19. A point moves in a plane with acceleration to a fixed point O according to the law μr. Prove that its motion resolved on any axis is simple harmonic of definite period, that its areal velocity about O is constant, and that the hodograph of the motion is the same as the orbit itself except in scale and phase.

20. When a circle rolls on a straight line at a uniform rate, show that the hodograph of any carried point is a circle uniformly described. How is the position of the pole of the hodograph related to the distance of the carried point from the centre of the rolling circle ?

21. The motion of a point in a plane is defined by $x = at^2$, $y = bt^{-1}$. Prove that the areal velocity is constant, and verify that the acceleration is everywhere directed from a fixed point according to the law $2r/t^2$.

22. The motion of a point in a plane is defined by $x = a \cos 2nt$, $y = 8\sqrt{2}a \cos nt$. Draw a graph of the complete path. Prove that the magnitude of the acceleration at any point is proportional to the distance of the point from a fixed line parallel to the axis of y.

23. Find the radius of curvature at any point of the curves :

(1) $y = \cos x$.

(2) $x = \sin 2t, \quad y = 4\sqrt{2} \sin t$.

(3) $r = a + b \sin \theta$.

24. The radius of curvature being a, a constant, prove that

$$r^2 = 2ap + b,$$

where r is the radius vector, p the perpendicular to the tangent from the fixed origin, and b is constant. Prove that this equation can only represent a circle.

25. A point moves along a straight rod with uniform speed u, while the rod rotates in a plane about a fixed point O, through which it passes, with uniform angular velocity n. Find the radius of curvature of the path of the point at a distance r from O.

26. If ϕ is the angle which the radius vector r makes with the tangent at its extremity, prove that the radius of curvature ρ is given by

$$\frac{1}{\rho} = \frac{1}{r} \sin \phi + \frac{d\phi}{dr} \cos \phi.$$

PART II

DYNAMICS OF TRANSLATION

CHAPTER IV

FORCE AND MOTION

89. In seeking to connect the motion of a material body with the conditions which determine it, it is natural and necessary, in the first instance, to consider a motion of very simple type and, with this view, to exclude the possibility of rotation. A pure translation of a rigid body of finite size implies that every point of the body shares in the same displacement; between any two points there is no relative motion in space. But in any case, if the dimensions of a body be small, the relative motion of its points will also be small even when the body possesses a given angular velocity. This leads to the conception of a material *particle*, endowed with the dynamical properties of matter but the geometrical properties of a point. As such, a particle possesses three degrees of freedom in space of three dimensions, but no capacity for rotation. An ordinary finite body will behave as a particle if all causes tending to produce rotation are either absent or neutralized by suitable constraints, which deprive the body of the corresponding degrees of freedom. Thus the present part of the subject may be called the dynamics of a particle. But this must be taken to mean that the complications inherent in rotational motion are deliberately avoided, rather than that the common bodies which are the subject of experience are replaced by mere fictions.

90. The discovery of the principles of ordinary dynamics was the work of the seventeenth century. Kepler, who arrived in its early years at a fairly accurate kinematical description of the solar system, had no conception of them. Galileo laid the foundations,

and by the end of the century Newton had formulated a system which coordinates a vast range of phenomena in a satisfactory way and leaves for the future only the more precise definition of the limits of its application.

The principles on which the Newtonian system of dynamics is founded are embodied in three statements or laws, of which the first is:

Newton's first law*: Every body continues in its state of uniform velocity (including rest as a particular case) except so far as it is compelled to change that state of motion by external forces.

The effect of this law is to assert a point of view in opposition to historical misconceptions. It is common experience that motions started by human agency tend to stop, unless maintained by further effort. This is no longer to be regarded as in the nature of things, but as due to the action of resisting forces which need to be overcome. Nor is motion in a circle to be considered inherently natural or perfect, as the Greek philosophers held. On the contrary, uniform velocity is the natural state, deviations from which in respect of speed or direction are to be attributed to the action of forces.

This meaning of the law is quite clear and simple. What is not so easy is to define a state of rest or uniform velocity, for this involves a fixed frame of reference, and it has been seen that no such frame exists. It must suffice here to say that this difficulty is theoretical rather than practical, for in general it is possible to choose a frame of reference which, without possessing an absolute character, can be treated as such for the purpose of any given problem.

91. The property of matter in virtue of which a body resists any change in its state of motion is the *inertia* or *mass* of the body. It is measurable in terms of an appropriate unit, which is independent of the units of length and time. It is an intrinsic property of every body, and additive in the sense that, if m_1, m_2 are the masses of two bodies separately, m_1+m_2 is the mass of the two bodies taken together. As this rule is found to be unaffected by any degree of chemical fusion between the two bodies, mass may be regarded as the dynamical measure of the quantity of matter contained in a body.

Def. The *density* of a substance is its mass per unit volume. Thus, if m is the mass of a substance which fills a volume V with uniform density ρ,

$$\rho = m/V, \quad m = \rho V.$$

* Lex I. Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.

If the density is not uniform, the ratio m/V represents the average density in the volume V, and the actual density at any point is

$$\rho = \text{Lim.}\, \frac{m}{V},$$

where the limit is to be found by making the volume V surrounding the point, and the contained mass m, diminish indefinitely.

The unit of mass may be taken to be the mass of a particular platinum standard, known as the British imperial pound.

The dimensions of density are 1 in mass, and -1 in volume or -3 in length. With British units the unit of density is that of 1 cubic foot containing 1 lb. of matter uniformly distributed. Another unit of density is that of water under definite standard conditions. The density of any substance in terms of this unit is the *specific gravity* of the substance. Being the ratio of two densities it is a mere number without physical dimensions and is independent of the system of fundamental units adopted.

The particle is an ideal conception, because it is assumed to have mass without volume.

92. *Def.* The *momentum* (linear) of a moving body is the product of the mass of the body and its velocity.

Thus, if the mass is m lbs. and the velocity v ft./sec., the momentum of the body is M units of momentum, where

$$M \text{ units of momentum} = mv \text{ lb. ft./sec.}$$

The unit of momentum is chosen as the momentum of unit mass moving with unit velocity, so that $M=1$ when $m=v=1$. or

$$1 \text{ unit of momentum} = 1 \text{ lb. ft./sec.}$$

With this consistent choice of units,

$$M = mv$$

expresses the relation between the corresponding numerical measures.

Now the velocity v is a vector, and m is an ordinary scalar multiplier. Hence the momentum M is also a vector, having the same direction as v and capable of being resolved into components according to the parallelogram law. Such components, in fact, are the corresponding components of v, multiplied by m.

93. Newton's second law*: The rate of change of momentum is proportional to the acting force, and the change is parallel to the direction of the force.

* Lex II. Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.

The primary function of this law is to provide a definition of force. The (time) rate of change of momentum, being proportional to the force, may be taken as the measure of the force. Thus, when the mass remains constant,

$$\begin{aligned}\text{force} &= \text{rate of change of momentum}\\ &= \text{mass} \times \text{rate of change of velocity}\\ &= \text{mass} \times \text{acceleration};\end{aligned}$$

and if F units of force are required to produce an acceleration f ft./sec.^{2} in a mass of m lbs.,

$$F \text{ units of force} = mf \text{ lb. ft./sec.}^2.$$

The consistent unit of force will make $F=1$ when $m=f=1$, or is that force which produces an acceleration of 1 ft./sec.^{2} in 1 lb., so that

$$1 \text{ unit of force} = 1 \text{ lb. ft./sec.}^2.$$

With this unit, which is called the *poundal*,

$$F=mf, \quad f=F/m,$$

express the relation between the corresponding numerical measures.

Now the acceleration f is a vector, and m is a scalar multiplier. Hence the force F is also a vector with the same direction as f, and its components are equal to the corresponding components of f, multiplied by m. Thus any component of F has the same effect in the acceleration of the mass m as it would have if the other components were absent. If (X, Y) are the components of F parallel to two axes, and $(\ddot{x}, \ddot{y})$ are the components of f parallel to the same axes,

$$X=m\ddot{x}, \quad Y=m\ddot{y},$$

and a third similar equation is needed when the motion is in three dimensions.

94. Expressed immediately in symbols, Newton's second law takes the form

$$\dot{M}=\frac{d}{dt}(mv)=F,$$

where the constancy of m need not be assumed. As the rate of change of a vector is also a vector, the vectorial character of the force F follows, as before, from this definition. Though masses will here be regarded as constant, this more general form of statement is of fundamental importance.

To define force as the rate of change of momentum, and to measure it by the acceleration it is capable of producing in a given mass, may seem to need some justification. The sensation of

muscular exertion leads to an idea of force, especially in static conditions, which may not seem in harmony with this definition. An example may be instructive.

A vertical cylinder is filled with gas and closed by a smooth piston. The gas is retained under pressure by a weight on the piston. If the weight is replaced by the pressure of the hand there is no real difference in the circumstances. The pressure, which causes the sensation of muscular exertion, is derived from the weight of the body, and to that extent reduces the pressure of the body on the ground.

In this case the gas exerts on the piston a palpable pressure, measured by the weight required to maintain equilibrium. But the real action of the gas, though resulting in an apparent force, can be expressed in terms of momentum and time. The piston is bombarded by molecules which have their momentum (in the direction of the axis of the cylinder) reversed on hitting the piston. The pressure is nothing else than the effect of this bombardment, and it is the same thing as the time rate at which the piston changes the momentum of the molecules of the gas.

On the other hand, the action of the weight, which appears simple only because it is familiar, is far less easy to understand. The weight may be said to result from the gravitational attraction of the earth on the mass used. But the nature of this " action at a distance " is the reverse of self-evident.

95. A force, as defined, is a finite vector with a definite direction acting at a point. This again is an ideal conception, like the conception of a material particle. For finite forces in nature do not act *at* points, but express the resultant of actions which are distributed over surfaces or volumes. Thus every part of a body possesses weight, and the single force which represents the weight of the body as a whole results from an action which extends to all parts of the volume occupied by the body. So, when two bodies touch one another, contact extends over a finite area, however small, and is never confined to an absolute point. The force exerted by one body on the other is the resultant of the pressure over an area of contact. The intensity of this action, or pressure, is measured by the force per unit area, and is essentially different in linear dimensions from force. Strictly the word " pressure " (and its opposite, " tension ") should be restricted to the intensity of an action on a surface. But the word may often be used conveniently

for whole pressure or resultant pressure, which is a force, when confusion is not to be feared.

The measurement of a force F by the acceleration produced in a mass m, through the relation

$$F=\dot{M}=m\dot{v},$$

involves the determination of the velocity v. This implies the choice of a definite frame of reference. It is enough to consider a single axis in the direction of v, with a definite origin O. If A is another point on this axis, independently chosen, so that $AO=x$, the velocity of O relative to A is $\dot{x}$ and the acceleration of O relative to A is $\ddot{x}$. Hence the acceleration of the mass m relative to A is $\ddot{x}+\dot{v}$. When the motion is referred to the frame with origin A, a new measure of the force will result, namely

$$F'=m(\ddot{x}+\dot{v})=F+m\ddot{x}.$$

Thus, in general, the determination of the force depends on the choice of the frame of reference. If, however, $\ddot{x}=0$, the same measure of the force will be obtained, and this will happen when the choice of the frame is made among all those which differ only by a uniform motion of translation. Such a motion will not affect the measurement of forces and cannot be detected in this way. This accords with the common experience that when a train is moving uniformly, relative motions within a railway carriage are the same as if the train were at rest. It also explains why the motion of the earth is not readily detected by ordinary dynamical experiments.

96. It has already been remarked that all bodies falling freely without hindrance at a given place on the earth's surface descend with the same vertical acceleration g, the value of g being approximately 32·2 ft./sec.^{2}. This fact was verified by Newton through a series of experiments with pendulums of different materials. The force which causes this common acceleration in all masses is their weight. Thus if W_1, W_2 are the weights of two masses m_1, m_2,

$$W_1=m_1g, \quad W_2=m_2g,$$

and therefore

$$W_2/W_1=m_2/m_1,$$

or the weights at a given place are proportional to the masses. This property gives the means of comparing masses, or expressing them in a standard unit, if their weights can be compared by means of a common balance or steelyard. The result of weighing with such an instrument gives the ratio of the masses, irrespective of the

value of g and therefore valid for all places. But a weight, which is a force, is by no means to be identified with the corresponding mass. For the same mass m weighed in two places by means of a spring balance calibrated for one value g will give the results

$$W = mg, \quad W' = mg',$$

and the apparent weights W, W' of the same body will vary as the accelerations g, g' at the two places.

Thus weight, when it has a direct meaning, is a (slightly) variable force according to the locality on the earth's surface. It clearly has no such meaning when applied, for example, to the planet Mars. But a planet has a definite mass, though this can only be measured by methods which belong to celestial mechanics.

97. The weight of one pound, being the force which produces an acceleration g ft./sec.^{2} in a mass of one pound, is equal to g poundals. If it is taken as a unit of force, a standard value must be assigned to g, such as 32·2.

The value of g, apart from slight local variations, varies systematically with the latitude, being greatest at the poles and least at the equator. This systematic variation is due to two causes, (*a*) the figure of the earth, which is not spherical, and (*b*) the rotation of the earth. The latter cause is the more effective, and its influence is easily estimated.

Ex. To find the reduction in the value of g at the equator, due to the rotation of the earth.

For a rough estimate, the equator is 25,000 miles in length, or $25{,}000 \times 5280 = 1{\cdot}32 \times 10^8$ ft. There are 86,400 secs. in a solar day, or $8{\cdot}62 \times 10^4$ secs. in a sidereal day, the true period of rotation. Hence the speed of a point on the equator is $v = 1{\cdot}53 \times 10^3$ ft./sec. The angular velocity of rotation about the earth's axis is

$$n = 2\pi/8{\cdot}62 \times 10^{-4} = 7{\cdot}3 \times 10^{-5}/\text{sec.}$$

Hence the acceleration of a point on the equator towards the centre of the earth is

$$\begin{aligned} n^2 a = nv &= 7{\cdot}3 \times 1{\cdot}53 \times 10^{-2} \\ &= 0{\cdot}112 \text{ ft./sec.}^2 \\ &= g/290 \text{ roughly.} \end{aligned}$$

This acceleration, though imperceptible, requires force in the same direction to produce it. Thus, if $W = mg$ would be the weight of a mass m in the absence of the rotation, the fraction $mg/290$ is effective for this purpose, and only the balance, or roughly $289mg/290$, will represent the measurable weight of m at the equator.

98. To some extent the present conception of force must be allowed to become familiar through the consideration of its application to a series of concrete problems. It is based on the assumption that space possesses the properties defined by Euclidian geometry, and that time is an independently measured quantity. It attributes to an identified portion of matter, so long as it is unaltered by such effects as evaporation, condensation or the loss or gain of subatomic elements (*e.g.* electrons), an invariable mass. Then the equation

$$F = \frac{d}{dt}(mv)$$

will always define a force appropriate to any motion. On this system, without further appeal to theory or experience, the universe will present a connexion of forces which may, or may not, be simple and convenient. As a matter of fact the resulting scheme is found to possess these qualities over a wide range of phenomena.

But this, the classical or Newtonian system of dynamics, is not the only conceivable system. In attempting to extend the range of application to phenomena not originally contemplated, it has been found more convenient, in the sense of providing a simpler force system, to admit that inertia has a definite dependence on velocity, while force and momentum are still related by the above equation. In itself this involves no contradiction of the classical system, though it offers an alternative with special advantages for certain purposes. It is only when Euclidian geometry and the corresponding kinematical scheme are themselves abandoned that the Newtonian system becomes necessarily inadequate.

When force is identified with the rate of change of momentum and so defined, this may seem to confuse cause with effect. But those words belong to the language of philosophy, not of science. It is the function of dynamics not to explain motion in any ultimate sense, but to predict the motions which will follow in given circumstances. With this object forces are introduced as a connecting link in the chain of phenomena. As measured by their dynamical effects their usefulness results partly from the uniformity of nature. By this principle, for example, the weight of a terrestrial body is a force so far invariable that it is the same whether the body is falling freely, or supported at rest on a table, or moving under constraints which attach it to other bodies. But there is another principle, founded on experience, which is of fundamental importance.

99. Newton's third law*: Corresponding to every action there is always an equal and opposite reaction; or between two bodies acting on one another, the action of the one is equal in magnitude and opposite in direction to the action of the other.

The first two laws of motion refer to the translation of a single rigid body, subject to external forces which are impressed on the body from outside. When these forces are assigned the motion can be deduced.

But when two or more bodies, subject to assigned external forces, are connected together, internal forces are developed within the system. These forces are in general unknown and require to be calculated in addition to the motions. They are of various kinds, but according to the law of action and reaction occur always as pairs of equal and opposed forces. Such a pair, exemplified by the pressure between two bodies in contact or the tension of a string, constitutes a *stress*.

It has been remarked that a finite force acting at a point represents the resultant of an action which always extends over a finite surface or volume. At every point of the action the intensity is equal and opposed, so that the two elements of a stress are not merely equal in total effect but are also identically distributed.

When a system of bodies is completely isolated from the influence of external forces, as the bodies constituting the solar system may be considered to be, the only forces possible take the form of internal stresses. External forces are themselves only the unbalanced elements of stresses, which must be treated as such in considering a partial system from which other bodies are arbitrarily excluded although they exert an influence. Thus, when two bodies A and B are connected by a taut string, there is at every point of the string a stress (tension). If the string be imagined to be cut at any point, the body A will continue to move as before if a force equal to the tension be applied at the point of section and treated as an external force. Actions at a distance follow the same law as actions by contact. Thus the attraction of the earth, which constitutes the weight of a body and causes it to fall, is only one part of a stress. An equal and opposite force acts on the earth and urges it in the direction of the falling body, with negligible effect on account of the overwhelming mass of the earth.

* Lex III. Actioni contrariam semper et aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi.

Contact stresses, whether in the nature of pressure or tension, are evoked with the intensity required to maintain the contact or geometrical connexion. The only limit is set by the strength of the materials concerned.

100. As a first application, the motion of a particle, of mass m, on a smooth inclined plane under gravity will be considered.

A smooth surface implies that no tangential stress is possible; the stress R between the particle and the plane is at right angles to the plane. The other force acting on the particle is its weight mg.

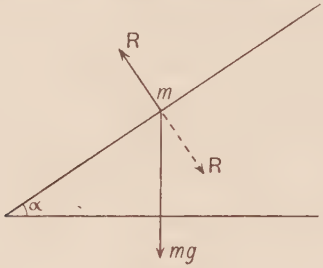

FIG. 50.

If α is the inclination of the plane, the components of this force are $mg\cos\alpha$ perpendicular to the plane and $mg\sin\alpha$ down the line of greatest slope. The particle remaining on the plane, there is no acceleration normal to the plane; let f be the acceleration down it. The relation of force to acceleration in these two directions gives

$$R - mg\cos\alpha = 0,$$
$$mg\sin\alpha = mf,$$

or $f = g\sin\alpha$. If the particle starts from rest, after a time t it will acquire the velocity

$$v = ft = g\sin\alpha \,.\, t$$

in travelling a distance down the plane

$$s = \tfrac{1}{2}ft^2 = \tfrac{1}{2}g\sin\alpha \,.\, t^2.$$

The corresponding fall, measured in the vertical direction, is $h = s\sin\alpha$. Hence

$$v^2 = 2g\sin\alpha \,.\, s = 2gh,$$

which is the same result as would follow if the particle fell freely through the same vertical distance. Thus the velocity acquired depends only on the vertical fall, and not otherwise on the inclination of the line of descent.

101. The time of descent, being given by

$$\tfrac{1}{2}gt^2 = s/\sin\alpha,$$

is the same for all lines, for which $s/\sin\alpha$ is constant. Let $AC=d$ be a vertical line, on which the circle $ABCD$ is described as diameter, AE being the horizontal tangent. If

$$EAB=ACB=\alpha, \quad AB=AC\,.\,\sin\alpha,$$

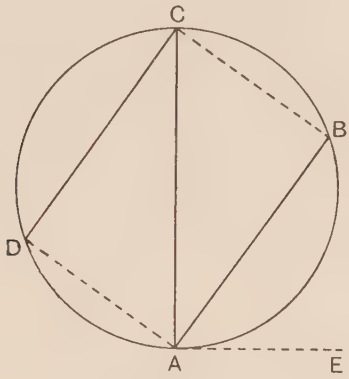

FIG. 51.

since $ABC=90°$. Hence the time of descent on the chord BA, or the equal parallel chord CD, is given by

$$\tfrac{1}{2}gt^2=AB/\sin\alpha=AC=d,$$

and is the same for all chords ending at A or starting from C, the lowest or highest points of the circle.

This being true for all the circles of which AC is the vertical diameter, it follows that the time of descent down all chords of a sphere, which pass through the highest or lowest points, is the same and equal to the time down the vertical diameter. It also follows, by taking an oblique section of this sphere through its highest or lowest points, that the proposition first proved for a vertical circle is equally true for the chords of a circle in an inclined plane.

An interesting application of this principle is afforded by the problem of finding the straight line of quickest descent in given

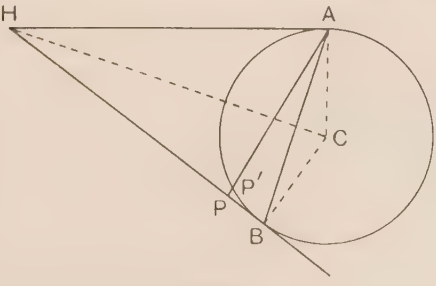

FIG. 52.

conditions. Thus, to find this line in a vertical plane from a given point A to a line HB, let H be the point where the given line meets the horizontal line HA through A. Let $HB=HA$; then AB will be the line required. For a circle can be drawn touching HA at A and HB at B. Since A is its highest point, if P is any point on HB except B, AP will cut the circle in P' such that the time down AB is equal to the time down AP', which is clearly less than the time down AP.

102. The more general conditions, which determine the line of quickest descent from one curve to another in the same vertical plane, can be considered in the following way. Let $CEDF$ be a

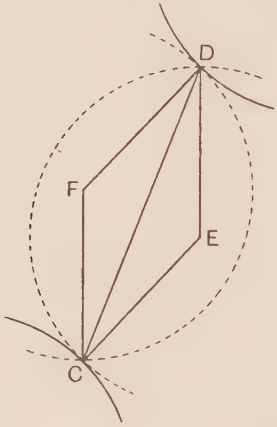

FIG. 53.

rhombus, drawn with its sides CF, DE vertical. Since $EC=ED$, the point D in the figure will be the highest point of the circle drawn through C with the centre E. Hence DC will clearly be the line of quickest descent from D to any curve which has exterior contact with this circle at C. Similarly, since $FC=FD$, the point C in the figure will be the lowest point of the circle drawn through D with the centre F. Hence DC will be the line of quicker descent to C than a line drawn to C from any point on a curve which has exterior contact with this circle at D. Since the time of descent will thus be increased by varying the position of C or D on either curve, DC will be the line of quickest descent from one curve to the other, as compared with any other connecting straight line. The points C and D on the two curves are therefore determined by these conditions:

(a) The normals to the curves, CE, DF, are parallel.

(*b*) These normals, together with the vertical lines through C and D, form a rhombus, so that the line CD at each end bisects the angle between the normal and the vertical.

(*c*) The curves must have external contact with the circles drawn with centres at E and F.

It is easy to see that if the condition (*c*) is reversed, and the curves have internal contact with the two circles defined, the line CD satisfying the condition (*b*) will be a line of slower descent than by any neighbouring line connecting the curves.

Ex. To find the line of quickest descent from a given line to a circle in the same vertical plane.

Let the diameter BC, through the centre O, meet the given line DN

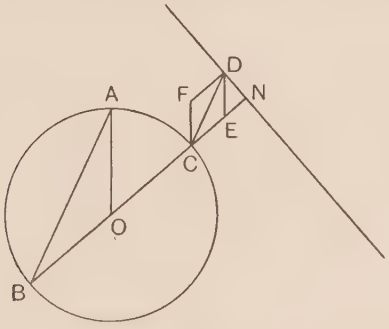

FIG. 54.

at right angles in N. Let OA be the vertical radius of the circle. Then CD drawn parallel to AB is the required line.

Draw DE, CF vertical, or parallel to OA, and DF parallel to BC or perpendicular to DN. The triangles AOB, DEC, CFD, having parallel sides, are similar. Therefore $ED = EC$, and $FC = FD$, since $OA = OB$. Hence the parallelogram $CEDF$ is a rhombus, with its sides CE, DF normal respectively to the circle and the given straight line, and its sides CF, DE vertical. Hence the diagonal CD satisfies the necessary conditions, and is the line of quickest descent from the straight line to the circle.

103. The application of the same fundamental dynamical principles will be further illustrated in the following examples, in which the conditions are such as to produce uniform accelerations, and the resulting motion is therefore of the simplest type.

Ex. A particle of mass m is placed on a smooth inclined plane, which is the surface of a body of mass M, capable of moving freely in the horizontal direction. To determine the motion of the system.

Let h be the horizontal acceleration of the mass M, and f the acceleration of the mass m down the inclined surface. If α is the inclination of

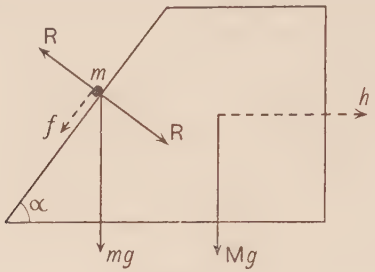

FIG. 55.

this surface to the horizon, the components of the acceleration of m relative to the ground are $h - f\cos\alpha$ (horizontal), $f\sin\alpha$ (vertically downward). Let R represent the equal action and reaction between the particle m and the body M. The components of the forces acting on m are

$$-R\sin\alpha \text{ (horizontal)}, \quad mg - R\cos\alpha \text{ (vertically downward)},$$

while the horizontal component of R acting on M is $R\sin\alpha$. The relations between corresponding components of force and acceleration now give the equations

$$m(h - f\cos\alpha) = -R\sin\alpha,$$

$$mf\sin\alpha = mg - R\cos\alpha,$$

$$Mh = R\sin\alpha,$$

which are sufficient in number to determine f, h and R. Thus, on eliminating R,

$$(M+m)h - mf\cos\alpha = 0,$$

$$Mh\cos\alpha + mf\sin^2\alpha = mg\sin\alpha,$$

and, on eliminating h,

$$mf\{(M+m)\sin^2\alpha + M\cos^2\alpha\} = m(M+m)g\sin\alpha.$$

Hence

$$f = \frac{M+m}{M+m\sin^2\alpha}\cdot g\sin\alpha,$$

$$h = \frac{m}{M+m\sin^2\alpha}\cdot g\sin\alpha\cos\alpha,$$

and

$$R = \frac{Mm}{M+m\sin^2\alpha}\cdot g\cos\alpha.$$

The vertical force acting on the mass M is $Mg + R\cos\alpha$. This, however, produces no effect, being neutralized by the upward pressure

of the ground or an equivalent constraint. The measure of this reaction is therefore

$$Mg + R\cos\alpha = Mg\left(1 + \frac{m\cos^2\alpha}{M + m\sin^2\alpha}\right)$$
$$= \frac{M(M+m)}{M + m\sin^2\alpha}\,.\,g.$$

104. *Ex.* The circumstances of the previous example are so far changed that the particle m (a motor car, for example) ascends the inclined plane with the given acceleration k.

The reaction R between the particle and the plane is no longer normal to the plane, but acts in a direction making the angle β with the

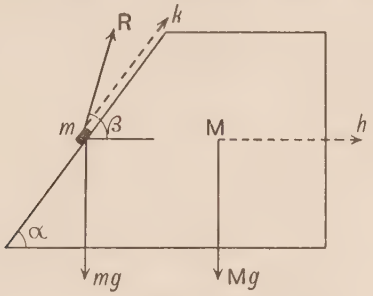

FIG. 56.

horizon. Here R, β and the horizontal acceleration h of the mass M are to be determined.

The components of force on the mass m are thus

$$R\cos\beta \text{ (horizontal)}, \quad R\sin\beta - mg \text{ (vertically upward)},$$

and the corresponding components of acceleration, relative to the ground, are

$$h + k\cos\alpha \text{ (horizontal)}, \quad k\sin\alpha \text{ (vertically upward)}.$$

Therefore
$$m(h + k\cos\alpha) = R\cos\beta,$$
$$mk\sin\alpha = R\sin\beta - mg.$$

The acceleration h of the mass M is due to the horizontal component, $-R\cos\beta$, of the reaction R, so that

$$Mh = -R\cos\beta.$$

By addition,
$$(M+m)h + mk\cos\alpha = 0,$$

or
$$h = -\frac{m}{M+m}\,.\,k\cos\alpha.$$

Hence the horizontal and vertical components of R are

$$R\cos\beta = \frac{Mm}{M+m}\,.\,k\cos\alpha,$$
$$R\sin\beta = m(k\sin\alpha + g),$$

giving R and β. The components of R up the plane and normal to it are

$$R\cos(\beta-\alpha)=\frac{m}{M+m}\{k(M+m\sin^2\alpha)+g(M+m)\sin\alpha\},$$

$$R\sin(\beta-\alpha)=\frac{m}{M+m}\{km\sin\alpha\cos\alpha+g(M+m)\cos\alpha\}.$$

In the preceding example there is no tangential component of R, or $\beta-\alpha=90°$, so that

$$0=k(M+m\sin^2\alpha)+g(M+m)\sin\alpha$$

$$-k=\frac{M+m}{M+m\sin^2\alpha}\cdot g\sin\alpha,$$

thus reproducing the value of f in that case. On the other hand, the normal pressure on the plane vanishes, or $R\sin(\beta-\alpha)=0$, if

$$-k=\frac{M+m}{m\sin\alpha}\cdot g,$$

and no greater acceleration down the plane is possible while the mass m remains in contact with the inclined surface of M.

105. *Ex.* A mass M is connected by a light string passing over a small fixed pulley to another light pulley, over which passes a second light string connecting two masses m_1, m_2. To find the accelerations of the masses and the tensions in the strings.

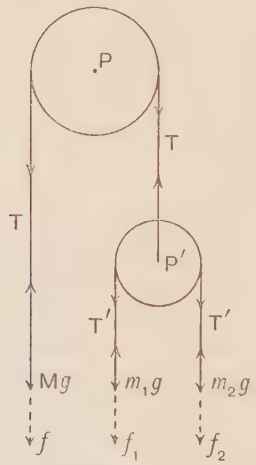

FIG. 57.

Let T, T' be the tensions in the two strings and f, f_1, f_2 the downward accelerations of the three masses M, m_1, m_2 respectively. Five equa-

tions are necessary to determine these quantities, and three are given directly by the motion of the three masses, namely :

$$Mf = Mg - T,$$
$$m_1 f_1 = m_1 g - T',$$
$$m_2 f_2 = m_2 g - T'.$$

The upward acceleration of the moving pulley P' is clearly f, and the downward accelerations of m_1, m_2 relative to P' are $f_1 + f$, $f_2 + f$. These being equal and opposite,

$$f_1 + f + f_2 + f = 0.$$

The upward force acting on the moving pulley is $T - 2T'$, but it is implied that this pulley has no mass (or momentum). Hence

$$T - 2T' = 0,$$

and this completes the required number of equations. From the first four,

$$g - m_1^{-1} T' + g - m_2^{-1} T' + 2(g - M^{-1} T) = 0,$$

or, since $T = 2T'$,

$$4g = (m_1^{-1} + m_2^{-1}) T' + 2M^{-1} T$$
$$= T' (m_1^{-1} + m_2^{-1} + 4M^{-1}).$$

The tensions are therefore

$$T' = \frac{4g}{4M^{-1} + m_1^{-1} + m_2^{-1}},$$

$$T = \frac{8g}{4M^{-1} + m_1^{-1} + m_2^{-1}},$$

and the accelerations are

$$f = g - M^{-1} T = \frac{-4M^{-1} + m_1^{-1} + m_2^{-1}}{4M^{-1} + m_1^{-1} + m_2^{-1}} \cdot g,$$

$$f_1 = g - m_1^{-1} T' = \frac{4M^{-1} - 3m_1^{-1} + m_2^{-1}}{4M^{-1} + m_1^{-1} + m_2^{-1}} \cdot g,$$

$$f_2 = g - m_2^{-1} T' = \frac{4M^{-1} + m_1^{-1} - 3m_2^{-1}}{4M^{-1} + m_1^{-1} + m_2^{-1}} \cdot g.$$

As a particular case, $f = 0$ and M will remain at rest if

$$\frac{4}{M} = \frac{1}{m_1} + \frac{1}{m_2},$$

and then

$$f_1 = -f_2 = \frac{m_1 - m_2}{m_1 + m_2} \cdot g.$$

On the other hand, if $M = m_1 + m_2$, the mass M will not remain at rest, but will have the acceleration

$$f = \frac{(m_2 - m_1)^2}{m_1^2 + m_2^2 + 6m_1 m_2} \cdot g,$$

which, being positive, is always downward.

106. When a body slides in contact with a smooth surface there is a reaction or stress which is purely normal to the surface. The absence of a component in any other direction is implied by the word *smooth*. When the body slides (without rolling) in contact with a rough surface, the normal reaction R is accompanied by a stress μR in the plane tangential to the surface. In this case, the only one to be considered at present, the laws of friction may be expressed in a simple form :

(1) The opposite components of the stress μR act in the line of relative motion, each in the sense or direction which opposes that motion.

(2) When motion is actually taking place the coefficient μ is a constant, depending only on the nature of the surfaces in contact.

The coefficient μ is called the coefficient of friction. It will sometimes be found convenient to represent it by $\tan\lambda$, in which case λ is called the angle of friction. Further consideration will be given to the subject of friction in connexion with statical problems.

Ex. A particle is projected directly up a rough plane, at inclination α, with the velocity V_1, and returns to its starting point, after coming to rest, with the velocity V_2. To find the coefficient of friction.

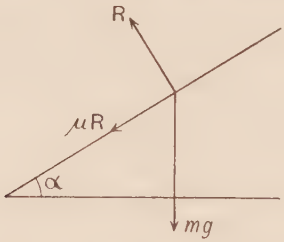

FIG. 58.

Let μ be the coefficient of friction, m the mass of particle, f_1 and f_2 the accelerations down the plane on the upward and downward path. In going upward, μR acts on the particle down the plane, and

$$R = mg\cos\alpha,$$

$$mf_1 = mg\sin\alpha + \mu R,$$

so that

$$f_1 = g(\sin\alpha + \mu\cos\alpha).$$

In the downward path the force μR is reversed, and therefore similarly

$$f_2 = g(\sin\alpha - \mu\cos\alpha).$$

But, if s is the distance travelled on the plane,

$$V_1^2 = 2f_1 s, \quad V_2^2 = 2f_2 s,$$

or $$\frac{V_1^2}{V_2^2}=\frac{f_1}{f_2}=\frac{\tan\alpha+\mu}{\tan\alpha-\mu};$$

and thus $$\mu=\frac{V_1^2-V_2^2}{V_1^2+V_2^2}\tan\alpha$$

is the required coefficient of friction.

107. The equivalence of force to rate of change of momentum is illustrated in the following example.

Ex. A machine gun weighing 28 lbs. fires 1 oz. bullets at the rate of 300 a minute with a velocity 1000 ft./sec. If the gun is mounted on a slope, which it is free to descend, what is the angle of the slope if the gun maintains its position when firing down it ?

Momentum is imparted to the bullets at the rate

$$\frac{1}{16}\cdot\frac{300}{60}\cdot 1000=\frac{10{,}000}{32}\text{ lb. ft./sec.}^2$$

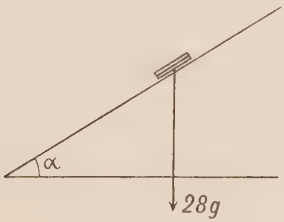

FIG. 59.

This is the measure of a force, and an equal reaction up the slope applies to the gun. The component of the weight down the slope is $28g\sin\alpha$ lb. ft./sec^2. Hence equilibrium results when

$$10{,}000/32=28g\sin\alpha,$$
$$\operatorname{cosec}\alpha=0{\cdot}0896g=2{\cdot}885,$$

or $\alpha=20°{\cdot}3$ when $g=32{\cdot}2$.

The action, here treated as continuous, is really discontinuous and can be analysed. The bullets are discharged at intervals of 0·2 sec. During a discharge (a very short time) the forces between gun and bullet are equal and opposite and, acting for the same time, produce equal and opposite momenta. The corresponding velocities are therefore inversely proportional to the masses. Hence at each discharge the gun gains a velocity $u=1000/(16\times28)$ ft./sec. up the slope. Being mounted freely, the gun has an acceleration $f=g\sin\alpha$ down the slope all the time, giving in 0·2 sec. a velocity $0{\cdot}2\,g\sin\alpha$. Since this must be neutralized at each discharge,

$$1000/(16\times28)=0{\cdot}2g\sin\alpha,$$
$$g\sin\alpha\quad=10{,}000/(32\times28),$$

as before.

Let the gun start from rest at a time 0·1 sec. before the next discharge. In 0·1 sec. it will gain the velocity

$$0{\cdot}1\, g \sin\alpha = 1000/(32\times 28) \text{ ft./sec.} = \tfrac{1}{2}u$$

down the slope, and it will have travelled the distance

$$\tfrac{1}{2}(0{\cdot}1)^2\, g \sin\alpha = 100/(64\times 28) = 0{\cdot}0558 \text{ ft.},$$

or $\frac{2}{3}$ in. nearly. The velocity is now reversed by the discharge of a bullet, and in the next 0·1 sec. the gun is brought to rest at its starting place, and the whole process is repeated.

108. In the examples considered the dynamical conditions are unaltered by a limited displacement of the parts of the system, and therefore constant forces give rise to constant accelerations. Hence the velocities and displacements which follow from given initial conditions can be traced.

The only other case in which methods have been given here for a similar solution is that of simple harmonic motion, in which the acceleration is proportional to the displacement reversed. The force which gives rise to this type of motion in a constant mass must also vary as the displacement and act in the opposite direction to it.

Ex. A cylinder floats in a liquid with its axis vertical and submerged to a depth b. The force of buoyancy being proportional to the volume submerged, to find the period of the vertical oscillations when the cylinder is disturbed.

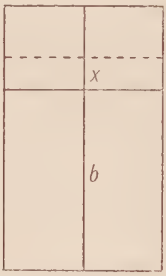

FIG. 60.

Let m be the mass of the cylinder. When the base is submerged to the depth $b+x$, the volume submerged is proportional to $b+x$ and the fluid pressure is represented by $\lambda(b+x)$, where λ is constant. In the position of equilibrium $x=0$ and

$$\lambda b = mg.$$

In the displaced position $\ddot{x}$ is the acceleration downward, and

$$m\ddot{x} = mg - \lambda(b+x)$$
$$= -\lambda x = -mgx/b,$$

or

$$\ddot{x} + gx/b = 0,$$

an equation of the typical form in simple harmonic motion.

Hence the displacement and velocity are given by

$$x = a \sin \sqrt{\frac{g}{b}}\,(t - t_0),$$

$$\dot{x} = a \sqrt{\frac{g}{b}} \cos \sqrt{\frac{g}{b}}\,(t - t_0),$$

where the amplitude a and the instant t_0 are arbitrary, and the period of oscillation is therefore

$$T = 2\pi \sqrt{\frac{b}{g}}.$$

109. The same type of motion is realized approximately by the simple pendulum, consisting of a particle connected to a fixed point by a weightless string, when slightly disturbed from the vertical position of equilibrium.

Let O be the fixed point, a the length of the string and m the mass of the particle. When disturbed in a vertical plane through

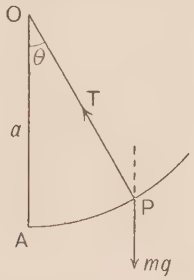

FIG. 61.

OA, the position of equilibrium, the particle moves on the arc of a circle AP with centre O and radius a. Let θ be the angle AOP, $s = a\theta$ the arc AP, and T the tension in the string OP. The motion being circular, the components of acceleration of m are $a\dot{\theta}^2$ along PO and $\ddot{s} = a\ddot{\theta}$ along the tangent at P or at right angles to PO. The components of the forces in these two directions give the equations

$$ma\dot{\theta}^2 = T - mg \cos \theta,$$
$$ma\ddot{\theta} = -mg \sin \theta.$$

The first of these will give T when $\dot{\theta}$ has been deduced from the second. But the latter is not in a form which has been discussed. It may, however, be written in the form

$$\ddot{\theta} = -\frac{g}{a}\theta \cdot \frac{\sin \theta}{\theta} \to -\frac{g}{a}\theta,$$

as $\theta \to 0$. Indeed, $\sin \theta/\theta$ differs little from 1 (but always in defect) so long as θ is a small angle, a necessary assumption.

The motion in small oscillations is therefore represented approximately by a familiar type of equation, and the solution is

$$\theta = \alpha \sin \sqrt{\frac{g}{a}}(t - t_0),$$

$$\dot{\theta} = \alpha \sqrt{\frac{g}{a}} \cos \sqrt{\frac{g}{a}}(t - t_0).$$

The period of a complete swing is thus

$$P = 2\pi \sqrt{\frac{a}{g}}.$$

The amplitude α is arbitrary, but must be small to justify the approximate method of solution. Since $\sin\theta/\theta < 1$, the value of g is effectively reduced by this factor, and the period is correspondingly increased. Hence P is the minimum period to which the pendulum approaches as the amplitude is indefinitely reduced.

110. What is commonly called a seconds pendulum is one which performs its swing in either direction in one second. Its period in the sense here defined is therefore 2 secs. Since

$$P^2 . g = 4\pi^2 . a,$$

with $P = 2$, $g = \pi^2 a$, so that corresponding values of g and a are

$$g = 32{\cdot}2 \text{ ft./sec.}^2, \qquad a = 3{\cdot}26 \text{ ft.}$$
$$= 981 \text{ cm./sec.}^2, \qquad = 99{\cdot}4 \text{ cm.}$$

Here a is the length of the simple pendulum beating seconds. Such a pendulum, in which the whole mass is concentrated in a single point, cannot be realized in practice. It serves as a model, however, with which real vibrating systems can be usefully compared.

The motion of the rigid pendulum will be considered in a later chapter. But one deduction can be made from the simple consideration of dimensions. Any one of a number of pendulums all geometrically similar can be distinguished by a length a measured according to some appropriate rule. The period of this member of the set will depend naturally on a and also on the value of the acceleration g, which has the dimensions LT^{-2}. But the only combination of a and g which is independent of length is a/g, which has the dimensions T^2. Hence the period, being a time, can only be expressed in terms of a and g alone by the form $\sqrt{(a/g)}$, multiplied by a factor which is a pure number without physical dimensions. This factor will depend on the geometrical form of the pendulum and the distribution of its mass.

111. The angular displacement and velocity, θ and $\dot{\theta}$, of the simple pendulum have been expressed in terms of the time t, but only by a method which is clearly approximate. But the accurate relation between $\dot{\theta}$ and θ is easily found, though the expression of either in terms of t is far from simple. By the result found in § 57,

$$\ddot{\theta}=\tfrac{1}{2}\frac{d}{d\theta}(\dot{\theta}^2)=\frac{-g}{a}\sin\theta,$$

$$\tfrac{1}{2}\dot{\theta}^2=\frac{-g}{a}\int\sin\theta\,d\theta=\frac{g}{a}\cos\theta+C.$$

At the end of the swing $\dot{\theta}=0$ and $\theta=\alpha$, the amplitude. Hence

$$0=\frac{g}{a}\cos\alpha+C$$

and

$$\tfrac{1}{2}\dot{\theta}^2=\frac{g}{a}(\cos\theta-\cos\alpha).$$

The tension of the string in any position can now be calculated accurately. For

$$\begin{aligned}T&=ma\dot{\theta}^2+mg\cos\theta\\&=2mg(\cos\theta-\cos\alpha)+mg\cos\theta\\&=mg(3\cos\theta-2\cos\alpha).\end{aligned}$$

Let V be the horizontal velocity of the bob through the position of equilibrium, and v the velocity in any position θ. Then

$$\begin{aligned}v^2&=a^2\dot{\theta}^2=2ag(\cos\theta-\cos\alpha),\\V^2&=2ag(1-\cos\alpha)=4ag\sin^2\tfrac{1}{2}\alpha,\\T&=\frac{m}{a}(3ag\cos\theta-2ag+V^2).\end{aligned}$$

If $V^2<4ag$, a real value of the amplitude α is obtained; if $V^2>4ag$, no such value is possible, and the motion is not oscillatory. In the latter case the bob tends to describe a complete circle in the vertical plane. This it would do if the string were replaced by a rigid weightless rod capable of bearing a thrust (T negative).

With a string, however, this is not possible. When $T=0$ the string ceases to be taut and the motion of the bob is freed from constraint. This happens when

$$\cos\theta=\frac{2ag-V^2}{3ag},\quad \sin^2\tfrac{1}{2}\theta=\frac{ag+V^2}{6ag}.$$

But

$$\sin^2\tfrac{1}{2}\alpha=\frac{V^2}{4ag}.$$

Hence $\alpha<\theta$ if $3V^2<2(ag+V^2)$ or $V^2<2ag$; the string remains taut till the end of the swing and an oscillation results with $\alpha<90°$.

If $V^2 > 2ag$, $\theta < \alpha$ and an oscillation with amplitude between 90° and 180° is impossible because the string becomes loose before the end of the swing. Finally, if $V^2 > 4ag$, continuous motion in a circle would set in, but does not do so unless $V^2 > 5ag$, when no value of θ can be found at which the tension $T = 0$.

112. The same form of vibrations will be found for a particle disturbed from its position of equilibrium at the bottom of a smooth bowl. The figure of the bowl may be taken to be that produced

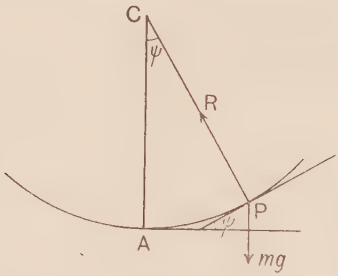

FIG. 62.

by rotating a curve about its normal AC; or the motion may be along a smooth wire in the vertical plane. The forces being mg, the weight of the particle, and R the normal reaction, the normal and tangential accelerations are determined by the equations

$$mv^2/\rho = R - mg\cos\psi,$$
$$m\ddot{s} = -mg\sin\psi,$$

where v is the velocity and ρ the radius of curvature at P, s is the arc AP and ψ is the angle between the tangents (or normals) at P and A. These equations are essentially of the same form as those found for the simple pendulum, the reaction R taking the place of the tension T. The second equation cannot be solved without a knowledge of the relation between s and ψ. Nevertheless, it may be written

$$\ddot{s} = -gs \cdot \frac{\sin\psi}{\psi} \cdot \frac{\psi}{s}$$
$$\to -gs/\rho, \quad \text{as } s \to 0,$$

where ρ is the radius of curvature at A. From this familiar form it follows that the period of small oscillations, when the particle is disturbed in a vertical plane, tends to the value

$$P = 2\pi\sqrt{(\rho/g)}$$

in the same way as the period of a simple pendulum has been represented.

113. In the case of a circle of radius a, the relation between s and ψ clearly takes the form $s=a\psi$. If now the curve were known for which $s=c\sin\psi$, the equation for motion along this curve becomes

$$\ddot{s}=-g\sin\psi=-gs/c,$$

which is precisely of the known form. It leads to

$$s=s_0\sin\sqrt{\frac{g}{c}}(t-t_0),$$

$$v=\dot{s}=s_0\sqrt{\frac{g}{c}}\cos\sqrt{\frac{g}{c}}(t-t_0),$$

and the period of oscillation with any finite arbitrary amplitude s_0 is

$$P=2\pi\sqrt{\frac{c}{g}}.$$

In fact, the time of descent from rest at any point to the lowest point of the curve is constant and equal to $\frac{1}{4}P$.

The curve which possesses this interesting property and is defined by the equation $s=c\sin\psi$ is the cycloid. Let ABD be the cycloid

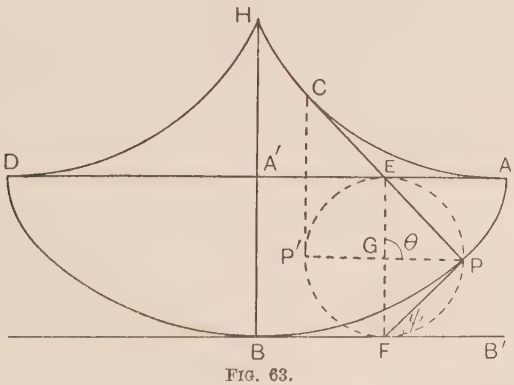

FIG. 63.

traced by the point P on a circle of radius a as it rolls along the line AD. The evolute is the curve AHD, which is an equal cycloid displaced in the plane, the part AH reproducing the part BD, and the part HD reproducing the part AB. Then

$$\text{arc } CA=CP=\rho=4a\sin\tfrac{1}{2}\theta,$$

where θ is the angle PGE through which the circle has rolled. The angle between CP, which is the tangent to CA, and the horizontal line AD is $\frac{1}{2}\theta$, while the angle between PF, the tangent to PB, and the horizontal line BB' is $\psi=90^\circ-\frac{1}{2}\theta$. Hence

$$s=\text{arc } BP=4a\cos\tfrac{1}{2}\theta=4a\sin\psi.$$

These relations (see § 22) show that the cycloid ABD is the curve defined by the equation $s=c\sin\psi$ when c is identified with $4a=BH$.

If the cycloidal support ABD be now removed, and the particle at P be attached to a light string HCP of length c and allowed to swing between two cycloidal guides HA and HD, it will move on the involute of AHD, and this is the cycloid ABD. In this form, discovered by Huygens, the pendulum will perform tautochronous oscillations, in which the period is independent of the amplitude.

114. The motion of a simple pendulum, or the similar motion of a particle in a smooth spherical bowl, has been limited to a vertical plane. This condition can be secured by projecting the bob or particle horizontally from its lowest position. In the vertical plane the particle has only one degree of freedom, expressed by the coordinate θ. If the particle be projected from any other point in an arbitrary direction at right angles to the string (or to the radius of the bowl), it will move with two degrees of freedom on a sphere. The problem in that case will lie beyond the present scope. There is, however, one case of great simplicity, in which the particle describes a circle in a horizontal plane with uniform angular velocity.

Let a be the radius of the circle and l the length of the string OB.

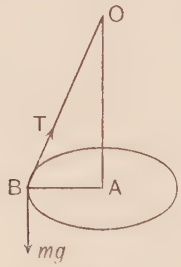

FIG. 64.

If θ is the constant inclination of the string to the vertical, $a=l\sin\theta$. Let n be the angular velocity of the particle, and T the tension of the string. Since the particle of mass m has the horizontal acceleration n^2a along the radius BA, and no vertical acceleration,

$$mn^2a=T\sin\theta,$$
$$0=mg-T\cos\theta,$$

and therefore $$T=mg\sec\theta, \quad n^2a=g\tan\theta.$$

Let $h=OA$, the depth of the circular path below the point of suspension, so that

$$h=l\cos\theta=a\cot\theta.$$

The period of rotation in the circle is thus

$$P=\frac{2\pi}{n}=2\pi\sqrt{\frac{h}{g}},$$

and depends only on the depth h.

This is an example of *steady motion*, in which, apart from the rotation, the geometrical and dynamical conditions remain constant. It is indeed possible to ignore the effect of motion, provided a force mn^2a acting on the particle in the direction of the outward radius AB is introduced to secure equilibrium against the horizontal component $T\sin\theta$. This fictitious force is called the *centrifugal force* due to the rotation. But it is more in harmony with the foregoing principles to say that the particle m, owing to its rotation, has the acceleration n^2a towards the centre of the circle, that this acceleration requires the force mn^2a to maintain it, and that this force is provided by the horizontal component of the tension T.

EXAMPLES.

1. Two masses M, m are connected by a string passing over a smooth light pulley. What vertical acceleration given to the pulley will make it possible for the mass M to remain at rest ?

2. Two smooth pulleys are fixed at opposite sides of a smooth table, and a string passes over them with masses m_1, m_2 hanging freely at its ends. A particle of mass M on the table between the pulleys is also attached to the string. Find the acceleration of the system when released, and the tensions of the string on either side of M.

3. Two masses m_1, m_2 are connected by a string passing over two light smooth pulleys. Between these pulleys the string is drawn down and passes under another light smooth pulley carrying a mass M. The parts of the string not in contact with the pulleys are vertical. The third pulley being released, the accelerations of m_1, m_2 are proportional to these masses. Prove that M rises with the acceleration $\frac{1}{2}g$ and express M in terms of m_1, m_2.

4. A triangular wedge of mass M, with smooth faces and parallel edges, is placed with one face on a horizontal plane. At the top of the other faces, inclined at angles α, β to the horizon, are placed particles with masses m_1, m_2 respectively. The wedge being free to move in the horizontal direction at right angles to its edges, the particles are allowed to fall freely. Prove that the acceleration of the wedge is

$$\tfrac{1}{2}g\cdot\frac{m_1\sin 2\alpha-m_2\sin 2\beta}{M+m_1\sin^2\alpha+m_2\sin^2\beta},$$

and that the particles reach the plane at the bottom simultaneously if

$$(M+m_1)\tan\alpha=(M+m_2)\tan\beta.$$

5. The centre of a circle in a vertical plane is C, and A is a fixed point on the vertical diameter. The lines joining any point V on the circle to C and A meet the circle again in P and Q. Prove that the time of free descent from rest under gravity down the chord PQ is constant and equal to the time of vertical fall through twice the distance CA.

6. Find the straight line of quickest descent under gravity to a given line from a circle in the same vertical plane.

7. Given two circles in a vertical plane, of which the lowest point of the one is on a higher level than the highest point of the other, find the straight lines of quickest and slowest descent under gravity from a point of the first to a point on the second.

8. A load M is raised through a vertical height h, starting from rest and coming to rest under gravity, by a chain only capable of supporting a stationary load $P(>M)$ with safety. Show that the shortest time in which this can be done is $\sqrt{\left(\frac{2h}{g}\cdot\frac{P}{P-M}\right)}$.

9. A large amount of material is to be raised from a given depth h by a chain for which the limiting safe load is P. Prove that the time occupied in the upward journeys from rest to rest will be the least possible if the material is sent up in loads of $\frac{2}{3}P$.

10. In sliding down a rough inclined plane under gravity the frictional resistance is $R\tan\lambda$, R being the normal reaction and λ constant. On a vertical line AB a circular arc containing the angle $90°+\lambda$ is constructed. Show that the time of descent of a particle from rest down any chord of this arc terminated at A or B, and treated as a line of greatest slope on the rough plane, is the same for all chords.

11. Two masses M_1, M_2 are connected by a string and placed on a line of greatest slope on a rough inclined plane, the inclination being α. The resistances to motion being respectively $\mu_1 R_1$ and $\mu_2 R_2$, where R_1, R_2 are the normal reactions, find the conditions necessary when the masses slide down the plane with the string taut, the common acceleration of the masses and the tension of the string.

12. Taking the formula

$$g=(978{\cdot}046+5{\cdot}184\sin^2\phi)\text{cm./sec.}^2,$$

where ϕ is the latitude, find the corresponding formula for the length of the seconds pendulum. In what latitude is the length 99·5 cm. ?

13. A pound weight is attached to a light wire a yard long, capable of supporting a stationary load of 1 cwt. If the wire is set in motion about one end so that the weight at the other describes a horizontal circle, find the greatest number of revolutions per minute which can be made safely, and the depth of the weight below the fixed point of support.

14. An endless chain of mass M and length c rotates in the form of a circle about its centre under no impressed forces. If P is the period of rotation, prove that the tension of the chain is Mc/P^2.

15. The bob of a simple pendulum of length a is projected from its lowest position with the horizontal velocity V, such that the string ceases to be taut after describing the angle θ. Prove that, if v is the velocity of the bob at the instant when the string becomes slack,

$$v^2 = -ag\cos\theta = \tfrac{1}{3}(V^2 - 2ag),$$

and that the centre of curvature of the parabolic path of the bob after this instant coincides initially with the point of suspension.

16. The bob of a simple pendulum of length a is projected from the position of equilibrium with the horizontal velocity $2\sqrt{(ag)}$. Prove that the string becomes slack when the bob has risen a vertical height $\frac{5}{3}a$ above its starting point, and that subsequently the bob passes at a height $\frac{2}{16}a$ directly over the point of suspension.

17. A particle is projected from the lowest point inside a smooth sphere of radius a with a velocity V such that $3{\cdot}5ag < V^2 < 5ag$. Prove that the parabolic path of the particle after leaving the sphere will pass through the point which divides the vertical diameter in the upward direction in the ratio

$$(2V^2 - 7ag)(ag + V^2)^2 : (2V^2 - ag)(5ag - V^2)^2.$$

18. A particle is projected from the lowest point A on the inside of a smooth sphere, and after describing a quadrant AB remains in contact with the surface until it has described a further angle ϕ ($<90°$). If the particle then returns to the sphere at the point D by a free parabolic path, prove that the arc BD subtends the angle 3ϕ, measured in the opposite direction to ϕ, at the centre of the sphere.

19. A particle projected from the lowest point on the inside of a smooth sphere leaves the surface after describing the angle θ about the centre ($90° < \theta < 180°$). In the subsequent parabolic path the particle travels at the inclination ψ to the horizon when crossing the vertical diameter of the sphere, and at the inclination χ when it reaches the sphere again. Prove that

$$\tan\psi = \tan^3\theta, \quad \tan\chi = 3\tan\theta.$$

20. Show that, if a heavy particle fall from the upper end down the complete arc of a smooth cycloid whose axis is vertical and vertex downwards, its pressure on the curve at any point is double that due to its weight alone.

21. A particle starts from the upper end of a complete cycloidal arc whose axis is vertical and vertex downwards. Show that it will fall through half the distance in arc towards the vertex in two-thirds of the time required to reach the vertex, and that three-quarters of its vertical descent will then have been accomplished.

22. A stream of fluid, of density ρ, meets directly a plane obstacle at the velocity v. If the fluid be assumed to flow outward at the rim parallel to the plate, show that the pressure on the obstacle is ρv^2 per unit area.

Calculate the pressure in pounds' weight per square foot due to a wind of 50 miles an hour, if a cubic foot of air weighs $1\frac{1}{4}$ oz.

23. Water passes through a pipe, one inch in diameter, at the rate of 125 gallons per minute. At one point the pipe is bent through a right angle. Show that the whole thrust of the water on the pipe at this point, due to the change in direction of the flow, is nearly equal to the weight of half a hundredweight.

24. The mass of a raindrop falling vertically is proportional to the time elapsing from its formation, when its mass and velocity are both zero. Prove that it descends with the acceleration $\frac{1}{2}g$.

25. A surface is formed by rotating the curve $y=f(x)$ about the axis of x, which is drawn vertically upward. On the smooth interior of this surface a particle describes a horizontal circle through (x, y) with uniform angular velocity n. Prove that

$$n^2 y \frac{dy}{dx} = g.$$

Hence show that the period is constant for circles at any height x if the surface is generated by a parabola rotated about its axis.

CHAPTER V

MOMENTUM. ENERGY. DIMENSIONS

115. It has now been shown by examples that the principles contained in Newton's laws can be applied to determine the motion not only of single particles but also of connected systems. Thanks to the law of equal action and reaction, the required number of equations can be formed to determine both the mutual stresses and the accelerations when the external forces are given. In this process each particle of the system has been treated separately. It is now to be seen that important results can be drawn from a consideration of the system as a whole, without treating the problem in all the detail which is otherwise necessary.

Let m be the mass of one particle of the system, and let f be the component of acceleration of this particle parallel to a certain direction defined by a line OX. Denote by X the corresponding component of the external forces, and by R_x the parallel component of the internal stresses, acting on m. Thus

$$mf = X + R_x,$$

and a similar equation applies to every particle of the system. The result of adding all these equations together may be written in the form

$$\Sigma mf = \Sigma X + \Sigma R_x.$$

But R_x is derived from a system of actions and reactions which occur in equal and opposite pairs, so that $\Sigma R_x = 0$ when the summation is extended to the whole system of particles. Hence

$$\Sigma mf = \Sigma X,$$

which is an equation independent of the internal stresses.

This is a notable extension of Newton's second law for a single particle, and may be thus expressed in words: in any chosen direction the rate of change of the momentum of any system is equal to the corresponding component of the total forces acting externally on the system. This principle is completely general.

116. If there are no external forces acting on the system, $\Sigma X = 0$. Then

$$\Sigma mf = \Sigma m\ddot{x} = 0,$$
$$\Sigma mv = \Sigma m\dot{x} = \text{const.},$$

or in words, the rate of change of the momentum in any direction being zero, the total momentum of the system remains constant. This is the principle of the *Conservation of Momentum*. It applies to any complete system in which the forces are confined to mutually balanced actions between different parts of the system.

The impossibility of communicating momentum to a material system as a whole without the intervention of forces from outside is a matter of universal experience. This is direct evidence for the truth of Newton's third law, that actions and reactions are equal and opposite.

Ex. Consider from this point of view the example of § 104. The horizontal acceleration of the mass M is h, and the horizontal component of the acceleration of m is $h + k\cos\alpha$. There is no external horizontal force acting on the system. Hence

$$Mh + m(h + k\cos\alpha) = 0,$$
$$h = -mk\cos\alpha/(M+m).$$

In the vertical direction, $k\sin\alpha$ is the upward acceleration of m and M has no acceleration. The external forces are the weights Mg and mg downwards, and the pressure P of the ground upwards.

Hence
$$mk\sin\alpha = P - Mg - mg,$$

or
$$P = (M+m)g + mk\sin\alpha$$

gives the equal pressure on the ground.

117. *Ex.* Two particles, of mass m_1, m_2, connected by a light inextensible string, move on a smooth table under no forces. The string

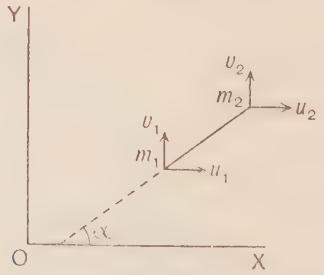

FIG. 65.

becomes taut when in the direction making the angle α with a fixed line OX. To determine the subsequent motion of the particles.

Parallel to the line OX and the perpendicular line OY, let (u_1, v_1) be the velocity components of m_1 and (u_2, v_2) those of m_2 before the string becomes taut, and let (u_1', v_1'), (u_2', v_2') be the corresponding components at the instant after. Since no external forces are acting, the momentum of the system is unchanged, and therefore

$$m_1u_1' + m_2u_2' = m_1u_1 + m_2u_2,$$
$$m_1v_1' + m_2v_2' = m_1v_1 + m_2v_2.$$

The taut string has no power to alter the momentum of either particle in the direction at right angles to itself, and therefore

$$u_1' \sin\alpha - v_1' \cos\alpha = u_1 \sin\alpha - v_1 \cos\alpha,$$
$$u_2' \sin\alpha - v_2' \cos\alpha = u_2 \sin\alpha - v_2 \cos\alpha.$$

Four equations have thus been found, but they are not independent, for they can be written

$$m_1(u_1' - u_1) = -m_2(u_2' - u_2) = m_1(v_1' - v_1)\cot\alpha,$$
$$m_1(v_1' - v_1) = -m_2(v_2' - v_2) = -m_2(u_2' - u_2)\tan\alpha,$$

by combining the first and third, and the second and fourth; or again,

$$\frac{u_1' - u_1}{m_2\cos\alpha} = -\frac{u_2' - u_2}{m_1\cos\alpha} = \frac{v_1' - v_1}{m_2\sin\alpha} = -\frac{v_2' - v_2}{m_1\sin\alpha} = \lambda.$$

A further equation is therefore necessary. But when the string has become taut, both particles move with the same velocity in the direction of the string, and this condition gives

$$u_1'\cos\alpha + v_1'\sin\alpha = u_2'\cos\alpha + v_2'\sin\alpha.$$

Hence
$$(u_1 + \lambda m_2\cos\alpha)\cos\alpha + (v_1 + \lambda m_2\sin\alpha)\sin\alpha$$
$$= (u_2 - \lambda m_1\cos\alpha)\cos\alpha + (v_2 - \lambda m_1\sin\alpha)\sin\alpha,$$

or
$$u_1\cos\alpha + v_1\sin\alpha + \lambda m_2 = u_2\cos\alpha + v_2\sin\alpha - \lambda m_1,$$

or
$$\lambda(m_1 + m_2) = (u_2 - u_1)\cos\alpha + (v_2 - v_1)\sin\alpha.$$

Thus, λ being now determined,

$$u_1' = u_1 + \frac{m_2\cos\alpha}{m_1 + m_2}[(u_2 - u_1)\cos\alpha + (v_2 - v_1)\sin\alpha]$$
$$= \frac{m_1 + m_2\sin^2\alpha}{m_1 + m_2}.u_1 + \frac{m_2\cos^2\alpha}{m_1 + m_2}.u_2 + \frac{m_2\sin\alpha\cos\alpha}{m_1 + m_2}(v_2 - v_1),$$

$$v_1' = v_1 + \frac{m_2\sin\alpha}{m_1 + m_2}[(u_2 - u_1)\cos\alpha + (v_2 - v_1)\sin\alpha)]$$
$$= \frac{m_2\sin\alpha\cos\alpha}{m_1 + m_2}(u_2 - u_1) + \frac{m_1 + m_2\cos^2\alpha}{m_1 + m_2}.v_1 + \frac{m_2\sin^2\alpha}{m_1 + m_2}.v_2;$$

and u_2', v_2' can be written down by interchanging the suffixes 1, 2.

The common velocity in the direction of the string immediately after it has become taut is therefore

$$u_1' \cos\alpha + v_1' \sin\alpha = u_2' \cos\alpha + v_2' \cos\alpha$$

$$= u_1 \cos\alpha + v_1 \sin\alpha + \frac{m_2}{m_1+m_2}[(u_2-u_1)\cos\alpha + (v_2-v_1)\sin\alpha]$$

$$= \frac{m_1u_1+m_2u_2}{m_1+m_2}\cos\alpha + \frac{m_1v_1+m_2v_2}{m_1+m_2}\sin\alpha,$$

while the unaltered velocities at right angles to the string have already been written down. The sudden change in the momentum of m_1 is thus in the direction of the string, and the amount of the change is

$$P = \frac{m_1m_2}{m_1+m_2}[(u_2-u_1)\cos\alpha + (v_2-v_1)\sin\alpha].$$

The corresponding change in the momentum of m_2 is also P along the string in the opposite direction.

118. This example has been worked out in formal detail as an instructive application of the principle of momentum. But the conditions assumed are artificial. Force, being measured by the rate of change of momentum, can only generate finite momentum in finite time. The sudden and discontinuous change P in this example implies an infinite tension in the string, which no string can bear without breaking. The action of the string in this case, though called very rapidly into play, must therefore be in reality gradual. For this and other reasons the inextensible string is a pure fiction.

Similar considerations apply in the case of an explosion. When a gun is fired the shot leaves the muzzle with a high velocity. For some purposes it may be legitimate to think only of the momentum given to the shot and the equal and opposite momentum given to the gun as though the action were instantaneous. In reality the shot travels with finite acceleration during the time of passage—perhaps 1/200 sec.—through the gun. The detailed history of what happens during this short interval of time constitutes the subject of Internal Ballistics.

Whenever a constraint affecting the motion of two bodies is suddenly introduced the case is similar. When one body impinges on another and appears to rebound immediately, the contact between the two really lasts for an appreciable time and is always accompanied by a deformation of the bodies. But the time of contact is very short, and the stresses evoked, though not infinite, are relatively very large. That they are large is shown by the

common experience that such actions often lead to the fracture of one of the bodies concerned. Thus a glass rod, capable of withstanding quite large stresses steadily applied, will readily break when dropped on the floor.

119. During the action of abnormal stress, the equation

$$mf = X + R_x$$

applies to the motion of the particle m at every instant. Let the whole duration of the action, t, be divided into small intervals δt, so that

$$m\Sigma f \,.\, \delta t = \Sigma X\, \delta t + \Sigma R_x\, \delta t,$$

the summation being extended to all the intervals which make up t. When the number of these intervals is increased indefinitely,

$$m\int f\, dt = \int X\, dt + \int R_x\, dt,$$

and therefore

$$m(v-u) = \bar{X} \,.\, t + \bar{R}_x \,.\, t,$$

where u and v are the initial and final velocity components of m and $\bar{X}$ and $\bar{R}_x$ are the mean values of the external force and reaction operating in the same direction during the time t. But during this short interval the force X is extremely small in comparison with R_x. Hence

$$m(v-u) = \bar{R}_x \,.\, t,$$

when $\bar{R}_x$ is the average value of a violent, short-lived force. But in general the course of development and the duration of this force are unknown. What is known is simply its effect, measured by the product $\bar{R}_x \,.\, t$ or P, where

$$P = m(v-u).$$

This quantity P is called an *impulse* and is measured by the change of momentum which it produces. The dimensions of an impulse are therefore the same as those of momentum (MLT^{-1}) and essentially different from those of a force (MLT^{-2}). Such expressions as "the force of a blow (or explosion)" for an impulse are misleading and unscientific, but the use of "impulsive pressure (or tension)," though strictly illogical, conveys a definite meaning and may be admitted.

120. The abstract reasoning employed above may be more easily understood when expressed in a graphical form. Let time be represented along the axis OT and forces by the ordinates parallel to OF. The impulse is developed in the short time t, represented by the section AB.

The component of external force X is relatively small, and may be considered constant in this interval. The action R_x is superposed

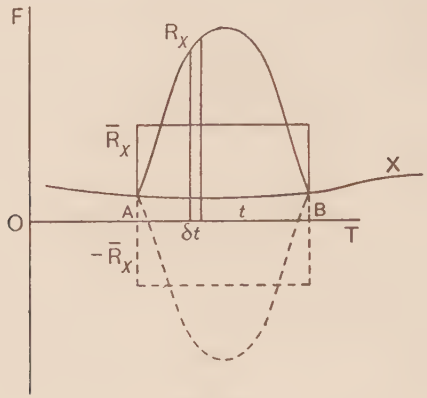

FIG. 66.

on this force and becomes relatively large. The impulse P has been given the forms

$$P=\int R_x\,dt=\bar{R}_x\,.\,t,$$

and is therefore represented by the area of the curve R_x above the curve X. It is equally represented by the area of a rectangle on the base t with the side $\bar{R}_x$, expressing the average value of R_x. The dynamical effect of the impulse depends on this area alone, or on the product $\bar{R}_x\,.\,t$, and not on the actual time t or on the shape of the curve R_x. For this reason it is possible to ignore the mode of development of the action, and to treat the effect as though it were produced suddenly.

To the action R_x corresponds an equal and opposite reaction $-R_x$, and therefore to the impulse P an equal and opposite impulse $-P$, represented by $-R_x t$. The law of action and reaction thus applies to impulses in the same sense as to forces.

Hence in a complete system, which includes all the particles affected by a set of impulsive stresses,

$$\Sigma m(v-u)=\Sigma P=0,$$
$$\Sigma mv=\Sigma mu,$$

or the momentum of the system as a whole is unaltered by the mutual impulses.

121. The simplest example of impulsive motion is presented when two spherical balls, of mass m_1, m_2, impinge directly on one

another. Let u_1, u_2 be the velocities before, and u_1', u_2' the velocities after, impact. Since the impact is direct, these velocities are in the line joining the centres and passing through the point of contact of the spheres. The momentum of the system is unaltered by the mutual impulse, and therefore

$$m_1u_1' + m_2u_2' = m_1u_1 + m_2u_2.$$

Another equation is required in order to determine u_1', u_2'. The simplest hypothesis that can be made in accordance with experience is that the relative velocity is reversed by the impact and reduced in a certain ratio, so that

$$u_1' - u_2' = -e(u_1 - u_2).$$

From a series of experiments Newton reached the conclusion that the value of e, now called the *coefficient of restitution*, is a constant depending on the nature of the materials in contact, and not on the masses or the relative velocities of the bodies. This experimental law may be accepted in ordinary cases when the relative velocities are not excessively large. The values of e range from 0·9 for glass on glass to zero for soft bodies which adhere.

In the present case

$$\begin{aligned}(m_1+m_2)u_1' &= m_1u_1 + m_2u_2 - em_2(u_1-u_2)\\ &= (m_1 - em_2)u_1 + (1+e)m_2u_2,\\ (m_1+m_2)u_2' &= m_1u_1 + m_2u_2 + em_1(u_1-u_2)\\ &= (1+e)m_1u_1 + (m_2 - em_1)u_2.\end{aligned}$$

The impulse on the ball m_1 when impact occurs is therefore

$$\begin{aligned}m_1u_1' - m_1u_1 &= \frac{m_1}{m_1+m_2}\{m_1u_1 + m_2u_2 - em_2(u_1-u_2) - (m_1+m_2)u_1\}\\ &= \frac{m_1m_2}{m_1+m_2}(u_2-u_1)(1+e),\end{aligned}$$

and that on the ball m_2 is equal and opposite to this.

When $m_1 = m_2$ and $e=1$, $u_1' = u_2$ and $u_2' = u_1$; or the result when two perfectly elastic spheres of equal mass meet directly is that their velocities are interchanged.

122. The same methods may be applied to solve the problem of oblique impact between two spheres, provided the spheres are smooth. This condition is necessary at present because a tangential action between the bodies would give rise to a motion of rotation, and this must be excluded.

Let U_1, U_2 be the velocities making the angles θ_1, θ_2 with the line of centres before impact. Let the corresponding velocities and

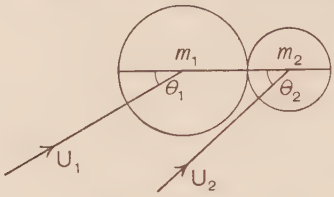

FIG. 67.

angles be U_1', U_2', ϑ_1', θ_2' after impact. If the components of these velocities along and perpendicular to the line of centres are (u_1, v_1), (u_2, v_2), (u_1', v_1'), (u_2', v_2'),

$$\begin{aligned} u_1 &= U_1 \cos\theta_1, & u_2 &= U_2 \cos\theta_2, \\ v_1 &= U_1 \sin\theta_1, & v_2 &= U_2 \sin\theta_2, \\ v_1' &= U_1' \sin\theta_1', & v_2' &= U_2' \sin\theta_2', \\ u_1' &= U_1' \cos\theta_1', & u_2' &= U_2' \cos\theta_2'. \end{aligned}$$

Because there is no tangential action the momentum of each sphere at right angles to the line of centres is unchanged ; hence

$$v_1' = v_1, \quad v_2' = v_2.$$

In the direction of the line of centres the conditions are the same as in the case of direct impact, so that

$$\begin{aligned} m_1u_1' + m_2u_2' &= m_1u_1 + m_2u_2, \\ u_1' - u_2' &= -e(u_1 - u_2). \end{aligned}$$

The equations written down make it possible to pass from U_1, U_2, θ_1, θ_2 to u_1, u_2, v_1, v_2, thence to u_1', u_2', v_1', v_2' and finally to U_1', U_2', θ_1', θ_2'. Thus the problem may be completely solved.

Ex. Two equal spheres meet symmetrically with equal velocities, the paths before impact being inclined to one another at the angle 2α, and after impact at the angle 2β. To find the coefficient of restitution.

In this case, U and U' being the equal velocities before and after impact,

$$\begin{aligned} u_1 &= U\sin\alpha = -u_2, \\ v_1 &= v_2 = U\cos\alpha = U'\cos\beta, \\ -u_1' &= U'\sin\beta = u_2', \end{aligned}$$

and

$$\begin{aligned} u_1' + u_2' &= u_1 + u_2, \\ u_1' - u_2' &= -e(u_1 - u_2). \end{aligned}$$

Thus

$$\begin{aligned} e &= -\frac{u_1' - u_2'}{u_1 - u_2} = \frac{2U'\sin\beta}{2U\sin\alpha} \\ &= \frac{\cos\alpha}{\cos\beta} \cdot \frac{\sin\beta}{\sin\alpha} = \tan\beta\cot\alpha. \end{aligned}$$

123. When the impact takes place on a smooth plane or the surface of a body which has a prescribed motion unaffected by the impact, the momentum of the system is altered by an external impulse. The condition of constant momentum is replaced by the condition $u_2' = u_2$, and the relation

$$u_1' - u_2' = -e(u_1 - u_2)$$

now becomes $$u_1' = -eu_1 + (1+e)u_2,$$

or, when the surface is fixed ($u_2 = 0$), simply

$$u_1' = -eu_1.$$

This refers to the velocity components in the direction of the normal to the surface at the point of impact. At right angles to the normal the components are unchanged, the surface being smooth.

Ex. A ball is dropped on the floor from a height h, the coefficient of restitution being e. To find the time and the distance traversed before the ball comes to rest.

The ball first reaches the floor with the velocity $u_1 = \sqrt{(2gh)}$ and rebounds with the velocity $u_1' = -e\sqrt{(2gh)}$, since $u_2 = 0$. The time of the first fall is therefore $u_1/g = \sqrt{(2h/g)}$, and the time of the following rise or fall is $-u_1'/g = e\sqrt{(2h/g)}$. The time for every successive fall is thus reduced by the factor e, and the whole duration of the motion is

$$\begin{aligned}&\sqrt{(2h/g)} + 2e\sqrt{(2h/g)} + 2e^2\sqrt{(2h/g)} + \dots \\ &= \sqrt{\left(\frac{2h}{g}\right)}\left[-1 + \frac{2}{1-e}\right] \\ &= \sqrt{\left(\frac{2h}{g}\right)} \cdot \frac{1+e}{1-e}.\end{aligned}$$

Since the velocity $e\sqrt{(2g/h)}$ is that due to a fall from the height e^2h, the height reached at each successive rebound is reduced by the factor e^2, and the whole distance traversed is therefore

$$h + 2e^2h + 2e^4h + \dots = -h + \frac{2h}{1-e^2} = h \cdot \frac{1+e^2}{1-e^2}.$$

The number of rebounds is infinite, but the ball comes to rest in a finite time, since $e < 1$.

124. *Ex.* A smooth spherical particle is projected from a point on an inclined plane. To find the condition that it shall rebound vertically at the first or any given subsequent impact.

Let α be the inclination of the plane, u the velocity of projection, making the angle θ with the line of greatest slope, and e the coefficient of restitution. It will be convenient to resolve the motion along and perpendicular to the plane; let x and y be the distances measured from O, the point of projection, in these directions.

The components of acceleration are

$$\ddot{x} = -g\sin\alpha, \quad \ddot{y} = -g\cos\alpha.$$

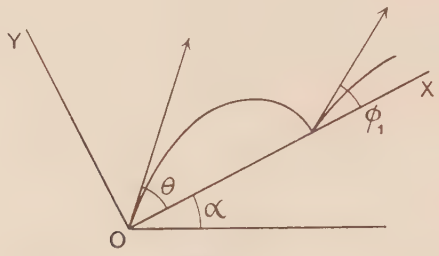

FIG. 68.

The components of velocity are initially $u\cos\theta$, $u\sin\theta$, and the displacements in time t are therefore

$$x = u\cos\theta \,.\, t - \tfrac{1}{2}g\sin\alpha \,.\, t^2,$$
$$y = u\sin\theta \,.\, t - \tfrac{1}{2}g\cos\alpha \,.\, t^2.$$

At the first impact,

$$y = 0, \quad t = 2u\sin\theta/g\cos\alpha,$$

and at the instant before impact the components of velocity are

$$\dot{x} = u\cos\theta - g\sin\alpha \,.\, t$$
$$= u(\cos\theta - 2\sin\theta\tan\alpha),$$
$$\dot{y} = u\sin\theta - g\cos\alpha \,.\, t = -u\sin\theta.$$

At the instant after impact these components become

$$v_1\cos\phi_1 = \dot{x} = u(\cos\theta - 2\sin\theta\tan\alpha),$$
$$v_1\sin\phi_1 = -e\dot{y} = eu\sin\theta,$$

where v_1 is the velocity of rebound, making the angle ϕ_1 with the line of greatest slope. Hence, on division,

$$e\cot\phi_1 = \cot\theta - 2\tan\alpha.$$

If the rebound is vertical, $\phi_1 + \alpha = 90°$. The condition for this is therefore

$$e\tan\alpha = \cot\theta - 2\tan\alpha,$$

or

$$\cot\theta = (2+e)\tan\alpha.$$

At the first rebound u, θ have become v_1, ϕ_1, with the conditions otherwise the same as at the moment of projection. At the next impact they are replaced by v_2, ϕ_2, and after n impacts by v_n, ϕ_n, where

$$e\cot\phi_1 = \cot\theta - 2\tan\alpha,$$
$$e\cot\phi_2 = \cot\phi_1 - 2\tan\alpha,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$e\cot\phi_n = \cot\phi_{n-1} - 2\tan\alpha.$$

These can be written in the form

$$2\tan\alpha=\cot\theta-e\cot\phi_1,$$
$$2e\tan\alpha=e\cot\phi_1-e^2\cot\phi_2,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$2e^{n-1}\tan\alpha=e^{n-1}\cot\phi_{n-1}-e^n\cot\phi_n,$$

and therefore, by addition,

$$2\tan\alpha(1+e+\dots+e^{n-1})=\cot\theta-e^n\cot\phi_n,$$

or
$$2\tan\alpha(1-e^n)=(1-e)\cot\theta-(1-e)e^n\cot\phi_n.$$

This equation determines ϕ_n in terms of α and θ. If the rebound is vertical after the n-th impact, $\phi_n+\alpha=90°$, and therefore

$$2\tan\alpha(1-e^n)=(1-e)\cot\theta-(1-e)e^n\tan\alpha.$$

Thus the required condition is

$$(1-e)\cot\theta=(2-e^n-e^{n+1})\tan\alpha,$$

which reproduces the first result when $n=1$.

125. When impact is a continuous process, the effect of an impulse is replaced by the effect of a force acting for a certain time.

Ex. A uniform flexible chain is suspended with the lower end in contact with a table. To find the pressure when the chain is released and is coiling itself on the table.

If ρ is the mass of the chain per unit length, ρx is the mass of a length x and $g\rho x$ the pressure due to this length coiled on the table.

At the moment when this length is coiled the vertical part of the chain has descended the same distance x under gravity, and has acquired the velocity $\sqrt{(2gx)}$. In a short time τ a further length $\tau\sqrt{(2gx)}$ will reach the table with this velocity, and its momentum,

$$\rho\tau\sqrt{(2gx)}\,.\,\sqrt{(2gx)}=2g\rho x\,.\,\tau,$$

is destroyed by contact with the table. The loss of momentum thus takes place at the rate $2g\rho x$. This is the measure of an upward force supplied by the table, and it is double the weight of the length of chain already coiled.

Hence the total pressure on the table at any time during the fall of the chain is $3g\rho x$, or three times the weight of the chain then lying on the table.

126. Consider a particle of mass m, moving in a definite direction under the constant force F, so that its acceleration is given by $mf=F$. If its velocity changes from u to v in a time t,

$$v-u=ft,$$
$$mv-mu=Ft,$$

and if a number of particles are connected in a system, in the sum

$$\Sigma mv-\Sigma mu=(\Sigma F)\,t.$$

If the forces vary with the time, instead of being constant, this equation must be written in the form

$$\Sigma mv - \Sigma mu = \int (\Sigma F)\, dt.$$

It has been seen that in the sum ΣF the internal actions do not occur. Further, since momentum is a vector, a similar equation applies to every direction. When $\Sigma F = 0$ for the external forces, an equation involving the velocities, but not the accelerations or the time, is obtained. The number of such distinct equations may be 1, 2 or 3 according to circumstances.

If, however, the motion of a single particle under a constant force F is considered from another point of view, its velocity changes from u to v in a distance s, where

$$v^2 - u^2 = 2fs,$$
$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = Fs.$$

When F, instead of being constant, varies along the path s the appropriate form of this equation (cf. § 57) is

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = \int F\, ds.$$

It is supposed here that the path s is straight and coincides in direction with F.

127. *Def.* The *kinetic energy* of a particle of mass m moving with the velocity v is measured by the product $\frac{1}{2}mv^2$.

Def. The *work* done by a force in a linear displacement of its point of application is measured by the product of the force into the component of the displacement in the same direction ; or (what is clearly the same thing) the product of the displacement by the corresponding component of the force.

The last equation above shows that a change of kinetic energy is equal to work done. The dimensions of work and energy are thus the same, namely ML^2T^{-2}, being in the one case force (MLT^{-2}) multiplied by a length (L) and in the other mass (M) multiplied by a squared velocity (L^2T^{-2}).

Unlike momentum, kinetic energy and work are not vectors. They will be found to be simply additive quantities.

One important property of work follows immediately from the definition. The work done by a force in any displacement is equal to the sum of the work done in the same displacement by any number of forces into which the original force may be resolved.

The forces OB, BC, CP, applied at one point, have the resultant OP. Their components in the direction of the displacement OA are

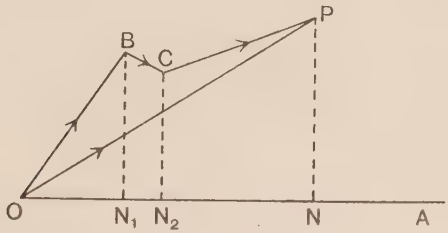

FIG. 69.

ON_1, N_1N_2, N_2N, and the sum of these components is equal to ON, the component of OP in the same direction. Thus

$$ON \,.\, OA = (ON_1 + N_1N_2 + N_2N) \,.\, OA,$$

or the work done by the resultant OP is equal to the sum of the work done by the components OB, BC, CP.

The work done by a force is negative when the displacement is in the opposite direction to the parallel component of the force. In this case the same work, taken positively, may be said to be done against the force. Thus, when a weight is hoisted, the work done against gravity varies jointly as the weight and the height to which it is raised. The definition therefore supplies a measure of work in accordance with the natural meaning of the term.

128. In general the force F acting on the particle m will not be constant, and the path of the particle will be a curve. Let δs be a

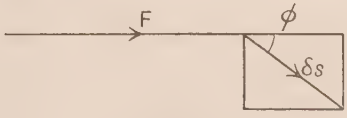

FIG. 70.

small arc of this curve, making the angle ϕ with the direction of F. In this direction the component of velocity changes from u to v, while the corresponding displacement is $\delta s \,.\, \cos\phi$. Hence, as in linear motion,

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = F \,.\, \delta s \,.\, \cos\phi.$$

In a direction at right angles to F the component of velocity w is unchanged in the interval, because there is no corresponding component of force. The last equation may therefore be written

$$\tfrac{1}{2}m(v^2 + w^2) - \tfrac{1}{2}m(u^2 + w^2) = F\cos\phi \,.\, \delta s,$$

where u^2+w^2 and v^2+w^2 are the total squared velocities of the particle at the beginning and end of the arc δs, which must be taken to be indefinitely small because F and ϕ are not constant. Thus the change in the kinetic energy is again equal to the work done on the particle, and the result of summing equations of this type over the successive elements of a finite arc will be

$$\tfrac{1}{2}mV^2-\tfrac{1}{2}mU^2=\int F\cos\phi\,.\,ds,$$

where U and V are the initial and final velocities of the particle, and the integral is written to represent the sum of the work done. In a system of particles an equation of this form applies to each, and the sum of all such equations may be written

$$\Sigma\tfrac{1}{2}mV^2-\Sigma\tfrac{1}{2}mU^2=\Sigma\int F\cos\phi\,.\,ds.$$

This expresses the result that the increase in the kinetic energy of the system is equal to the total work done by all the forces acting in the interval considered.

129. It has been seen that the work done by any force is equal to the sum of the work done by any components into which the force may be resolved. It is therefore unnecessary in calculating the total work done on the system to replace any separate forces acting on the particle m by their resultant F. It is, however, convenient to distinguish between external forces, which may be typified by P, and internal actions, which may be typified by R, and to write the general equation in the form

$$\Sigma\tfrac{1}{2}mV^2-\Sigma\tfrac{1}{2}mU^2=\Sigma\int P\cos\phi\,ds+\Sigma\int R\cos\theta\,.\,ds.$$

The nature of the integrals must be carefully understood, since they may differ essentially from those previously considered. These, as before, are limiting sums, but in general they cannot be obtained by a process which is the inverse of simple differentiation. When the curve s is known, $P\cos\phi$ will be known at every point of it, and the limiting sum can be calculated by some means, into which it is not necessary to enter. Since, however, the discovery of the curve s is usually the main object of the problem, this equation of energy cannot afford any help in the solution. In order that the equation may be practically useful in this sense it is necessary that the forces P shall be of such a kind that the corresponding integrals of work can be expressed in terms of the initial and final positions

alone, independently of the intervening path. When they are of this kind they are called *conservative* forces, because when a particle returns to its initial position by any re-entrant path the work done by the forces is in the sum zero and the kinetic energy is unaltered. The work done by any forces acting at a fixed point is zero, because there is no displacement. The work done by a smooth fixed constraint is also zero, because the action is normal to the surface and $\phi = 90°$.

Other considerations apply to the work done by the internal actions R of the system. When the equations of momentum are formed, these actions disappear always in the sum. In the equation of energy and work this may or may not be the case, according to the nature of the actions. In the sum $\Sigma R \cos\theta . \delta s$ there is always a component of reaction $-R\cos\theta$ corresponding to the component of action $R\cos\theta$. The action R therefore disappears from the equation when δs is the same for both these components, and not otherwise. Thus in the case of rigid connexions, illustrated by the thrust in a rigid rod or the tension in an inextensible string, the points of application of the components of a stress have a common motion, and the stress does not appear in the equation. But in the case of a tangential stress between two surfaces in relative motion at the point of contact, δs is not the same for both members of the stress and $\Sigma R\cos\theta . \delta s$ does not then vanish.

130. The equation of work and energy is a single equation applying to the system. It affords practical help towards finding the motion of the system only when the external forces are conservative and the internal actions are of such a kind that they do not occur in the equation. When this is the case, the equation of energy exists in the form

$$\Sigma \tfrac{1}{2} m V^2 - \Sigma \tfrac{1}{2} m U^2 = \Sigma \int P \cos\phi . ds$$
$$= W_1 - W_0,$$

where W_0 is a function of the initial positions of the particles and W_1 is a function of the final positions. This again may be written

$$\Sigma \tfrac{1}{2} m V^2 - W_1 = \Sigma \tfrac{1}{2} m U^2 - W_0 = \text{const.},$$

for, the two selected states of the system being arbitrary, the quantities relating to each, being equal, must have the same constant value throughout the motion. Here W_1, to which a constant may be added without affecting the form of the equation, may be

regarded as the work done by the forces in bringing the system to the state considered from any standard state it may be convenient to adopt for reference. Conversely, $-W_1$ is the work which the same forces would do if the motion were reversed and the system were restored from the given state to the standard state. Thus, if Q be written in the place of $-W_1$ and E for the kinetic energy $\Sigma\frac{1}{2}mV^2$,

$$E+Q=\text{const.}$$

Def. When a system is subject to a set of conservative forces, the work which the forces are capable of doing when the system in any state is reduced to a standard state or configuration is called the *potential energy* of the system.

The result expressed by the above equation may now be stated in the form: the sum of the kinetic and potential energies of a system remains constant during the motion of the system under conservative forces. Both these forms of energy are regarded as purely mechanical, and in the course of motion a simple interchange takes place between these two forms.

When *dissipative*, as opposed to conservative, forces are in action, mechanical energy as such is lost, being transformed into other forms recognized in physics. Potential energy then has no meaning as a physical quantity depending only on the geometrical configuration of the system, for the work done between one state and another depends not only on the two states but also on the intermediate states through which the system is made to pass.

131. The application of the equation of energy to systems moving under gravity is simple, because the weights are forces which are constant in magnitude and direction. Let δz be the projection

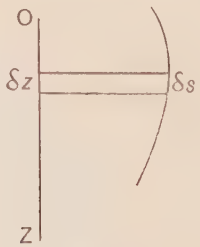

FIG. 71.

of an element δs of any arc on the fixed vertical line OZ, so that $\delta z=\delta s\,.\cos\phi$, where ϕ is the inclination of δs to OZ. Hence the work done by gravity when a particle of mass m descends the arc δs

is $mg \,.\, \delta s \,.\, \cos\phi = mg \,.\, \delta z$. The work done when the particle descends from the level of O to a depth z vertically below O is therefore mgz. The corresponding result for a system of particles may be expressed in the form $g\Sigma mz$. Accordingly, when the system is free from actions which contribute to the work done, the equation of energy becomes

$$\Sigma \tfrac{1}{2} m V^2 = g\Sigma mz + \text{const.}$$

The potential energy of the system is $-g\Sigma mz$, increased by any constant it may be convenient to introduce. If h is the height of the particle m, measured vertically upwards from a fixed origin O', while z is the depth measured downwards from O, $z+h$ is constant. Hence the potential energy of the system may be expressed by $g\Sigma mh$, this being the work which can be obtained from the action of gravity when the masses are lowered to the horizontal plane passing through the fixed point O'. In terms of the potential energy, the equation of energy becomes

$$\Sigma \tfrac{1}{2} m V^2 + g\Sigma mh = \text{const.}$$

In the case of a single particle moving under gravity with smooth constraints the equation of energy is

$$\tfrac{1}{2} m v^2 = mgz + \text{const.}$$

or, when z is the vertical depth below that level at which $v=0$,

$$v^2 = 2gz.$$

This expresses the general result which has already been noticed (§ 100) in the particular case of a particle descending a smooth inclined plane. Here the constraint may take the form of a smooth guide of any shape.

132. *Ex.* A particle is placed at the highest point of a fixed smooth

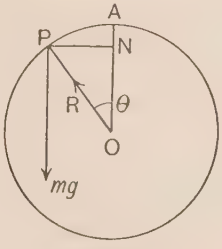

FIG. 72.

vertical circle, and, being disturbed, slides down on the outside of the circle. To consider the motion.

Let OA be the vertical radius, OP the radius which makes the angle θ with it, and N the foot of the perpendicular from P on OA. Let m be the mass of the particle, v its velocity on reaching P, R the normal pressure at this point and a the radius of the circle. From the equation of energy,

$$v^2 = 2g \,.\, AN = 2g(OA - ON)$$
$$= 2ga(1 - \cos\theta).$$

The acceleration of the particle along PO being v^2/a,

$$mv^2/a = mg\cos\theta - R.$$

Hence
$$R = m(g\cos\theta - v^2/a)$$
$$= mg(3\cos\theta - 2).$$

So long as the particle, starting from rest at A, remains in contact with the circle, R is positive. When R changes sign, which happens when $\cos\theta = \frac{2}{3}$, the particle leaves the circle and then moves freely as a projectile. At this instant $AN = a(1 - \cos\theta) = \frac{1}{3}a$, and $v^2 = \frac{2}{3}ga$. The particle therefore remains on the circle until it has descended a vertical distance equal to one-third of the radius, and then moves freely in a parabolic path, of which the initial conditions have been determined.

133. In the case of the simple pendulum, the tension of the string does no work, and the equation of energy takes the form

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}ma^2\dot{\theta}^2 = mg \,.\, a\cos\theta + \text{const.},$$

where a is the length of the string, m the mass of the bob, v its linear and $\dot{\theta}$ its angular velocity, and θ the angular displacement

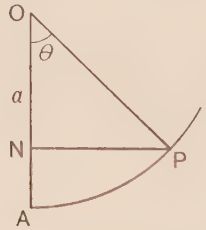

FIG. 73.

from the vertical. Here the work done is determined by the depth ON of the bob below the fixed point of suspension O. If α is the amplitude of the swing, so that $v = \dot{\theta} = 0$ when $\theta = \alpha$,

$$0 = mg \,.\, a\cos\alpha + \text{const.},$$

where the constant is the same as before. Hence, by subtraction,

$$\tfrac{1}{2}v^2 = \tfrac{1}{2}a^2\dot{\theta}^2 = ga(\cos\theta - \cos\alpha).$$

On the other hand, if V is the horizontal velocity of projection from the lowest position A, with the same constant,

$$\tfrac{1}{2}mV^2 = mga + \text{const.},$$

since $\theta=0$ in this position. Hence

$$\tfrac{1}{2}(V^2-v^2)=ga(1-\cos\theta).$$

These equations have already been obtained in substance (§ 111) by direct integration, and have been used to discriminate between the different types of motion which are possible. They are now seen to follow immediately from considerations of energy.

134. The approximate period for a simple pendulum

$$P=2\pi\sqrt{\left(\frac{a}{g}\right)}$$

is obtained from the simple harmonic motion

$$\theta=\alpha\sin\sqrt{\left(\frac{g}{a}\right)}(t-t_0),$$

$$\dot\theta=\alpha\sqrt{\left(\frac{g}{a}\right)}\cos\sqrt{\left(\frac{g}{a}\right)}(t-t_0),$$

to which the motion of the pendulum approaches, when the amplitude is very small. A second approximation to the period will now be found.

From these equations, which lead to the above value of P,

$$\dot\theta^2=\frac{g}{a}(\alpha^2-\theta^2),$$

whereas the true equation is

$$\dot\theta^2=\frac{2g}{a}(\cos\theta-\cos\alpha)$$

$$=\frac{2g}{a}\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{24}-\ldots-1+\frac{\alpha^2}{2}-\frac{\alpha^4}{24}+\ldots\right)$$

$$=\frac{g}{a}(\alpha^2-\theta^2)\{1-\tfrac{1}{12}(\alpha^2+\theta^2)+\ldots\}.$$

Hence a better value for P would result if in its expression a value g' were substituted for g such that

$$g'=g\{1-\tfrac{1}{12}(\alpha^2+\theta^2)\}.$$

Now θ varies through the motion, but $g'<g(1-\frac{1}{12}\alpha^2)$, and a better approximation will be found by assigning a mean value to θ^2. But in the first approximation

$$\theta^2=\tfrac{1}{2}\alpha^2-\tfrac{1}{2}\alpha^2\cos 2\sqrt{\left(\frac{g}{a}\right)}(t-t_0),$$

whence the mean value of θ^2 over a complete swing is $\frac{1}{2}\alpha^2$, the remaining term oscillating through positive and negative values equally. The appropriate mean value of g' is therefore

$$g' = g\{1 - \tfrac{1}{1\,2}\alpha^2(1+\tfrac{1}{2})\}$$
$$= g(1 - \tfrac{1}{8}\alpha^2).$$

When the pendulum swings with finite amplitude a, a second approximation to the period is therefore given by

$$P = 2\pi \sqrt{\left\{\frac{a}{g\left(1 - \frac{1}{8}\alpha^2\right)}\right\}},$$

which is greater than the limiting value.

135. In the application of the principle of energy it has been assumed not only that the forces are conservative, a condition which is satisfied by gravity, but also that the motion is not interrupted by one of those sudden actions which imply the intervention of an impulse. Since an impulse represents the effect of unanalysed forces acting for a short time it must not be assumed that an action of this kind will leave the constant of energy unaltered. On the contrary, energy is always changed in these circumstances, in form if not in total amount.

In the case of the direct impact of two spheres the change of energy is readily calculated. In the notation already used (§ 121)

$$m_1 u_1' + m_2 u_2' = m_1 u_1 + m_2 u_2,$$
$$u_1' - u_2' = -e(u_1 - u_2),$$

or, on squaring both sides,

$$m_1^2 u_1'^2 + m_2^2 u_2'^2 + 2m_1 m_2 u_1' u_2' = m_1^2 u_1^2 + m_2^2 u_2^2 + 2m_1 m_2 u_1 u_2,$$
$$m_1 m_2 (u_1'^2 + u_2'^2 - 2u_1' u_2') = m_1 m_2 (u_1^2 + u_2^2 - 2u_1 u_2) e^2.$$

By addition,

$$(m_1 + m_2)(m_1 u_1'^2 + m_2 u_2'^2) = (m_1 + m_2)(m_1 u_1^2 + m_2 u_2^2)$$
$$+ m_1 m_2 (u_1^2 + u_2^2 - 2u_1 u_2)(e^2 - 1),$$

and therefore the change in the kinetic energy of the system is

$$\tfrac{1}{2}(m_1 u_1'^2 + m_2 u_2'^2) - \tfrac{1}{2}(m_1 u_1^2 + m_2 u_2)$$
$$= -\frac{m_1 m_2}{2(m_1 + m_2)} (u_1 - u_2)^2 (1 - e^2).$$

There would thus be no change if $e = 1$. Since $e < 1$ there is always a loss of kinetic energy on impact.

136. The conservation of energy in its most general sense is a

fundamental doctrine of physics. In mechanics the principle of energy has a much more restricted application on account of the facility with which energy changes its form from that mode with which a purely mechanical problem is directly concerned. Just as the liability to fracture when one body comes into collision with another is simple evidence that abnormally large forces are called into play, so the click which is heard when one billiard ball strikes another is audible evidence that a wave motion has been set up in the material of the balls and communicated to the air. The waves require energy, which is drawn from the kinetic energy of the masses before impact and partly restored in the same form during the process of resilience. The residue of the energy, which in the form of waves is still purely mechanical but irrelevant to the visible motion of the masses, finally degenerates into the form of heat. The whole process of impact is therefore extremely complicated, and in the superficial view which has regard only to the palpable motions of the bodies it is necessary to admit an effective disappearance of energy.

The circumstances of two bodies connected by an inextensible string may also be considered, but in a simpler form than that already discussed in this chapter. Thus let two equal masses be thrown apart with equal and opposite velocities in the same straight line. When the connecting string, at first loose, becomes taut, the two masses are brought to rest, for the unchanged momentum of the system is zero. The kinetic energy of the system is at the same time destroyed, for if the string is truly inextensible and the masses are rigid, as supposed, no change with which energy can be associated is possible in the material of the system. There is no reservoir in which the energy which has disappeared can remain latent. As this destruction of energy contradicts the general physical principle, it must be inferred that inextensible strings and rigid bodies are fictions which have no real existence; and this conclusion has already been drawn above.

137. *Ex.* To find the work done in a time t from rest in the system considered in § 104.

Here a car, of mass m, ascends the inclined surface of a wedge, of mass M, free to move horizontally. The mass M has no vertical acceleration, but a horizontal component,

$$h = -\frac{mk\cos a}{M+m},$$

which is given immediately by the principle of momentum (§ 116).

The car has a vertical component of acceleration $k \sin\alpha$, and a horizontal component (relative to the ground),

$$h + k\cos\alpha = \frac{Mk\cos\alpha}{M+m}.$$

These components of acceleration are constant, and, when multiplied by t, give the rectangular components of velocity of M and m. Hence the kinetic energy of the system after a time t from rest is

$$\{\tfrac{1}{2}Mh^2 + \tfrac{1}{2}mk^2\sin^2\alpha + \tfrac{1}{2}m(h + k\cos\alpha)^2\}t^2$$

$$= \frac{1}{2}\cdot\frac{mt^2}{(M+m)^2}\{mMk^2\cos^2\alpha + (M+m)^2k^2\sin^2\alpha + M^2k^2\cos^2\alpha\}$$

$$= \frac{1}{2}\cdot\frac{m}{M+m}(M + m\sin^2\alpha)k^2t^2.$$

The car alone has a vertical component of acceleration, $k\sin\alpha$, and in the same time t ascends a vertical distance $\frac{1}{2}k\sin\alpha\,.\,t^2$. The work done in the system against gravity is therefore $\frac{1}{2}mgk\sin\alpha\,.\,t^2$.

Hence the total work done, partly against gravity and partly in producing the final kinetic energy of the system is

$$\tfrac{1}{2}mkt^2\left[g\sin\alpha + k\,.\,\frac{M + m\sin^2\alpha}{M+m}\right] = \tfrac{1}{2}kt^2\,.\,R\cos(\beta - \alpha),$$

where $R\cos(\beta - \alpha)$ is the tangential action between the inclined plane and the car (§ 104).

This work is derived from motor forces developed in the car. Since the wedge M has no vertical motion, its own weight and the pressure of the ground do no work. The work required to raise the mass m against gravity has been taken into account. The other forces in the system are the normal stress $\pm R\sin(\beta - \alpha)$ and the tangential stress $\pm R\cos(\beta - \alpha)$. The first of these does no work, because the displacement of car and wedge is the same in the direction at right angles to the inclined plane, so long as contact is maintained. But the tangential stress does do work, derived from the engine of the car. In the direction of this stress the car moves a distance s upwards in space, and the surface of the wedge a distance s' downwards. The relative displacement is $s + s' = \frac{1}{2}kt^2$, and the work done is

$$R\cos(\beta - \alpha)\,.\,s + R\cos(\beta - \alpha)\,.\,s' = R\cos(\beta - \alpha)\,.\,(s + s')$$
$$= R\cos(\beta - \alpha)\,.\,\tfrac{1}{2}kt^2,$$

which agrees with the above result and explains it.

138. *Def.* The *Power* developed in a system is the time rate at which work is done and energy is produced. If work and energy are produced uniformly to the amount W in the time t, the power

of the system is W/t. If the rate varies with the time, the power at any instant t is measured by

$$P = \text{Lim.}\frac{\delta W}{\delta t} = \frac{dW}{dt}.$$

The dimensions of work and energy being represented by ML^2T^{-2}, those of power are obtained by subtracting one in time, and are thus represented by the symbol ML^2T^{-3}.

Ex. In the example discussed in the preceding section, the work

$$\tfrac{1}{2}mkt^2\left[g\sin\alpha + k\,.\,\frac{M + m\sin^2\alpha}{M+m}\right]$$

is obtained from the engine of the car in the time t. The power developed by the engine at the instant t is therefore

$$mkt\left[g\sin\alpha + k\,.\,\frac{M + m\sin^2\alpha}{M+m}\right]$$

being derived from the work done in the same way as the velocity kt may be derived from a given displacement $\frac{1}{2}kt^2$.

139. When the point of application of a force F is displaced a small distance δs in a direction making an angle ϕ with the force, the work done by the force is

$$\delta W = F\cos\phi\,.\,\delta s$$

in a time δt. Hence the power of the agent which produces and maintains this force is

$$P = \text{Lim.}\frac{\delta W}{\delta t} = \text{Lim.}\left[F\cos\phi\,.\,\frac{\delta s}{\delta t}\right] = Fv\cos\phi,$$

where v is the velocity of the point of application of the force F. In words, therefore, the power corresponding to a force is measured by the product of the force and the component in the same direction of the velocity of its point of application.

When a particle of mass m is moving in a given straight line under a force F, its acceleration is given by $mf = F$, and $\phi = 0$. The power involved in the motion is thus

$$P = Fv = mfv.$$

If x is the displacement of the particle at the time t from a fixed point O on the line, and $\dot{x}$, $\ddot{x}$ are the corresponding velocity and acceleration, the power may be represented under the forms

$$P = m\dot{x}\ddot{x} = m\dot{x}^2\frac{d\dot{x}}{dx}.$$

140. When the velocity and acceleration of the particle are constant, the power is also constant. It may seem at first sight

that, when the velocity is constant, the acceleration and power must both be zero. But the term power is connected with an agent, or engine, maintaining a force and must be considered in relation to the circumstances in which this agent operates.

Ex. A load of mass m is hauled up a slope at the inclination α with uniform velocity v by a rope attached to a winding machine at the top of the slope. What power is necessary, when the surface offers no hindrance to the motion ?

The component $mg \sin \alpha$ of the weight acts directly down the slope. If the rope is given a tension $T = mg \sin \alpha$, this will balance the component due to gravity. The load can now remain stationary, and, as long as this is the case, no work is done and no power is used. But equally, under the balanced forces, the load can move with any assigned uniform velocity v along the slope. The function of the engine is thus to maintain the force T during the motion. As the point of attachment of the rope to the load now moves with the velocity v in the direction of the rope, the power taken from the engine is measured by

$$Tv = mgv \sin \alpha,$$

and it is easily verified that this is the rate at which work is done against gravity.

It is to be noticed that the power in this case is the power effectively used in raising the load from one level to another. An engine is a machine which converts energy in one form, *e.g.* the chemical energy of combustion, into other forms, *e.g.* mechanical work. In the process much of the original energy passes into forms which are not practically useful and may be positively harmful. When a definite object is aimed at, such as the raising of a weight or the production of an electrical current, only a fraction of the gross power of the engine is made available for the purpose in view. This fraction, or ratio of the effective power to the gross consumption of energy, which is invariably reduced in the course of transmission by any system of gears, represents the *efficiency* of the engine.

141. In what precedes it has been tacitly assumed that work, or energy, and power have been expressed in units consistent with the fundamental units of mass, length and time, whatever this fundamental system may be. If the British system, consisting of the mass of 1 lb., the foot and the second, be adopted, the corresponding unit of force is the poundal, expressed by 1 lb. ft./sec.^{2}. The resulting unit of work is the foot-poundal, being the work done by one poundal steadily applied through a displacement of one foot. Since work and energy are mutually convertible, the unit of energy (in any form) is also the foot-poundal, but since kinetic energy is represented by $\frac{1}{2}mv^2$, this unit may be associated with a mass of

two pounds moving with a velocity of 1 ft./sec. The foot-poundal, as the unit of work or energy, may be written 1 lb. ft.^{2}/sec.^{2}

In the same system the unit of power represents the power of an engine which does one unit of work per second. It may be written 1 lb. ft./sec.^{3} But these units of work and power are not in practical use. In physical science the C.G.S. system, with purely decimal modifications, is employed universally. In British technical practice the fundamental system is departed from at an early stage in the derived units.

Thus the British unit of force is the weight of one pound, or g poundals, since it produces an acceleration of g ft./sec.^{2} in a mass of one pound, where $g = 32{\cdot}2$. The corresponding unit of work or energy is the foot-pound, being the work done by gravity when a mass of 1 lb. is lowered through one foot, or the work done against gravity when the same mass is raised by the same amount.

The practical British unit of power is the horse-power, measured by 33,000 foot-pounds per minute or 550 foot-pounds per second. It is therefore $550g$ foot-poundals per second, or $550g$ lb. ft.^{2}/sec.^{3} The product $550g$ may be taken to be 17,700.

142. *Ex.* A hydro-electric turbine is operated by a fall through 5 ft. 6 in. of 6,200 cubic ft. of water per minute. Find the gross power available and the efficiency of the turbine when developing 50 H.P. at full load.

The mass of a cubic foot of water being 62·4 lbs., the mass delivered per second is

$$\frac{6{,}200 \times 62{\cdot}4}{60} = 62 \times 104 = 6448 \text{ lbs.}$$

The work done by gravity per second is therefore $6448 \times 5{\cdot}5 = 35{,}464$ foot-pounds, and the gross power is thus

$$\frac{35{,}464}{550} = 64{\cdot}48 \text{ H.P.}$$

Hence the efficiency when the turbine develops 50 H.P. is

$$\frac{50}{64{\cdot}48} = \frac{1}{1{\cdot}2896} = 0{\cdot}775,$$

or 77·5 per cent.

Ex. A particle of mass m has its velocity increased from u to v in passing over the distance s. Find the power at the beginning and end of the interval when the acceleration is constant, and the constant power which will have the same effect.

When the acceleration f is constant, it is given by

$$v^2 - u^2 = 2fs.$$

The power at the beginning of the interval is therefore

$$P_1 = mfu = mu(v^2 - u^2)/2s,$$

and at the end $$P_2 = mfv = mv(v^2 - u^2)/2s.$$

When the power P is constant,

$$P = m\dot{x}^2 \frac{d\dot{x}}{dx},$$

$$P \int dx = m \int \dot{x}^2 d\dot{x},$$

$$Px = \tfrac{1}{3} m\dot{x}^3 + \text{const.};$$

and since $\dot{x}$ changes from u to v in a displacement s,

$$Ps = \tfrac{1}{3} m(v^3 - u^3),$$

or $$P = m(v^3 - u^3)/3s.$$

Now $P_1/u = P_2/v$ and

$$\frac{P}{P_1} = \frac{v^2 + u^2 + uv}{u(v+u)} \cdot \frac{2}{3}.$$

Hence

$$P = \frac{2}{3} \cdot \frac{P_1^2 + P_2^2 + P_1 P_2}{P_1 + P_2}$$

$$= \frac{1}{2}(P_1 + P_2) + \frac{(P_2 - P_1)^2}{6(P_1 + P_2)},$$

which is greater than the arithmetic mean of P_1 and P_2.

143. In the C.G.S. system the fundamental units are the centimetre (length), the gramme (mass) and the mean solar second (time). The most important derived units are :

Force :	1 dyne	$= 1$ gr. cm./sec.^{2}
Work :	1 erg	$= 1$ gr. cm.^{2}/sec.^{2}
	1 joule	$= 10^7$ ergs.
Power :	1 erg per sec.	$= 1$ gr. cm.^{2}/sec.^{3}
	1 watt	$= 10^7$ ergs per sec.

The corresponding unit of momentum may be written 1 gr. cm./sec., but this has no particular name.

Any physical quantity requires two things for its specification, (1) a numerical measure and (2) an appropriate unit, which is itself generally a combination of other units considered more fundamental. Dynamical quantities involve units which combine the units of mass, length and time. If these are denoted by M, L, T, any dynamical quantity can be expressed in the form

$$n \,.\, M^\alpha L^\beta T^\gamma,$$

where n is a numerical measure, and α, β, γ are numbers which indicate the dimensions of the quantity in the respective fundamental units.

Let M_1, L_1, T_1 represent one system of fundamental units, and M_2, L_2, T_2 another system, and let n_1, n_2 be the corresponding measures of the same dynamical quantity. Then

$$n_1 . M_1^{\alpha} L_1^{\beta} T_1^{\gamma} = n_2 . M_2^{\alpha} L_2^{\beta} T_2^{\gamma}.$$

Hence the result of a transformation from the first system of units to the second is to change the numerical measure n_1 into n_2, where

$$n_2 = n_1 . \left[\frac{M_1}{M_2}\right]^{\alpha} \left[\frac{L_1}{L_2}\right]^{\beta} \left[\frac{T_1}{T_2}\right]^{\gamma},$$

the factor multiplying n_1 being a combination of mere ratios and therefore a pure number.

144. An excellent illustration, of obvious practical utility, is provided by the transformation from British units,

$$M_1 = 1 \text{ lb.}, \quad L_1 = 1 \text{ ft.}, \quad T_1 = 1 \text{ sec.},$$

to the C.G.S. system,

$$M_2 = 1 \text{ gr.}, \quad L_2 = 1 \text{ cm.}, \quad T_2 = 1 \text{ sec.}$$

Here

$$M_1/M_2 = 453{\cdot}59 = [2{\cdot}6567],$$
$$L_1/L_2 = 30{\cdot}4797 = [1{\cdot}4840],$$
$$T_1/T_2 = 1 = [0{\cdot}0000],$$

the number in square brackets being the common logarithm of the preceding number in each line.

The following table is easily constructed :

	Symbol.	$\log(n_2/n_1)$.	n_2/n_1.	
Velocity - -	LT^{-1}	1·4840	30·48	
Acceleration -	LT^{-2}	1·4840	30·48	
Momentum, Impulse -	MLT^{-1}	4·1407	13825	
Force - -	MLT^{-2}	4·1407	13825	= poundal/dyne
Energy, Work -	ML^2T^{-2}	5·6247	$4{\cdot}214 \times 10^5$	= ft. poundal/erg
Power - -	ML^2T^{-3}	5·6247	$4{\cdot}214 \times 10^5$	

The variability of the acceleration g with latitude has been mentioned. A mean value in C.G.S. units,

$$g_2 = 981 = [2{\cdot}9917] \text{ cm./sec.}^2$$

will be adopted. To this corresponds in British units

$$g_1 = 32{\cdot}19 = [1{\cdot}5077] \text{ ft./sec.}^2$$

Also

$$\log 550 = 2{\cdot}7404.$$

Hence, for the practical units of force, work and power,

1 lb. weight $= g_1$ poundals $= [5{\cdot}6484]$ dynes $= 4{\cdot}45 \times 10^5$ dynes.
1 ft.-lb. $= g_1$ ft.-poundals $= [7{\cdot}1324]$ ergs $= 1{\cdot}356$ joules.
1 H.P. $= 550 g_1$ ft.-poundals/sec. $= [9{\cdot}8728]$ ergs/sec.
$= 746{\cdot}2$ watts,

and thus measures in British units can be transformed into measures in the C.G.S. system.

145. *Ex.* What fundamental units will make the horse-power the unit of power, the foot-pound the unit of work, and acceleration under gravity the unit of acceleration ?

If M_1, L_1, T_1 be identified with the British system,

$$M_1 = 1 \text{ lb.}, \quad L_1 = 1 \text{ ft.}, \quad T_1 = 1 \text{ sec.}$$

The second system, M_2, L_2, T_2 is determined by the conditions

$$1 \,.\, M_2 L_2^2 T_2^{-3} = 550 g_1 \,.\, M_1 L_1^2 T_1^{-3},$$
$$1 \,.\, M_2 L_2^2 T_2^{-2} = \quad g_1 \,.\, M_1 L_1^2 T_1^{-2},$$
$$1 \,.\quad L_2 T_2^{-2} = \quad g_1 \,.\quad L_1 T_1^{-2}.$$

By successive division, $T_2^{-1} = 550\, T_1^{-1}$,

$$M_2 L_2 = M_1 L_1,$$

and therefore $L_2 = g_1 L_1 / 550^2 = [0{\cdot}0269 - 4] L_1$,

$$M_2 = 550^2 M_1 / g_1 = [3{\cdot}9731] M_1.$$

Hence $M_2 = 9400 M_1 = 9400$ lbs.,

$$L_2 = 1{\cdot}064 \times 10^{-4} L_1 = 1{\cdot}064 \times 10^{-4} \text{ ft.},$$
$$T_2 = T_1 / 550 = 0{\cdot}00182 \text{ sec.}$$

are the units required.

146. Any equation written down to express a physical relation may be regarded in its purely algebraic sense as a relation between measures, which are pure numbers ; or its several terms in the additive sense may be supposed to retain their physical dimensions. In the latter case, these dimensions must clearly be the same throughout. This principle, already noticed in particular applications, is of extreme value, for it yields information as to the form of the solution before a given problem has been fully answered, and even in cases where theory is unable to furnish a complete answer. Its greatest value in such instances is to guide experiment, so as to obtain results with the greatest economy of effort and to give such results the widest possible interpretation.

Ex. Closed orbits of any given possible shape are described by particles with an acceleration directed to a fixed point and varying as

the n-th power of the distance from it. To find the relation between the period and the scale of an orbit.

The scale of any orbit may be defined by any characteristic linear dimension a. The period in that orbit may be denoted by P. The acceleration, represented by μr^n, is of dimension LT^{-2}. Hence the constant μ has the dimensions $L^{1-n}T^{-2}$. It is implied in the problem that the time P depends only on a and μ. The only combination of a and μ which has no dimension in L is μa^{n-1}, and this has the dimensions T^{-2}. Hence the only possible relation connecting P with a and μ, and with no other physical quantity, is

$$\mu a^{n-1}P^2 = \text{const.}$$

Two cases are specially important. When $n=1$, the acceleration varies directly as the distance from the fixed point, and $\mu P^2 = \text{const.}$, or the period is independent of the scale of the orbit. When $n=-2$ the acceleration varies inversely as the square of the distance, and

$$\mu P^2/a^3 = \text{const.}$$

This expresses Kepler's third law for planetary orbits. But it should be noticed that in this reasoning there is nothing to exclude the possibility that the constant occurring in the relation may depend on the shape of the orbit; that the shape does not affect the statement of Kepler's law results from the choice of the major axis of the orbit as the characteristic linear dimension.

147. *Ex.* To consider the speed of capillary waves (ripples) on the surface of a liquid.

These waves are maintained by surface tension t, which is measured by a force acting per unit length across any section of the surface. The dimensions of t are therefore MLT^{-2}/L or MT^{-2}.

The other relevant conditions are the density ρ of the liquid, of dimensions ML^{-3}, and λ, the length of the waves. Since v, the speed of the waves, is independent of mass, it can only depend on t/ρ, of dimensions L^3T^{-2} and on λ. Since the dimensions of v are represented by LT^{-1}, those of v^2 are represented by L^2T^{-2}, which are also those of $t/\rho\lambda$. Hence it may be concluded that

$$v^2 \propto t/\rho\lambda,$$

because this is the only possible relation which can be formed with consistent dimensions.

The method is effective in this instance because it can be assumed that the three quantities t, ρ, λ are the only operative factors. Its weakness lies in the possibility that other factors may be overlooked. Thus in this example the effect of gravity has been ignored, and though this is justifiable it will be instructive to allow for the introduction of g. Let

$$g \propto t^\alpha \rho^\beta \lambda^\gamma,$$

a relation which is only consistent when

$$LT^{-2} = (MT^{-2})^\alpha (ML^{-3})^\beta L^\gamma.$$

From a comparison of the indices of M, L, T,

$$0 = \alpha + \beta,$$
$$1 = -3\beta + \gamma,$$
$$-2 = -2\alpha,$$

whence $$\alpha = 1, \quad \beta = -1, \quad \gamma = -2,$$

or $g \propto t/\rho\lambda^2$. Thus $g\rho\lambda^2/t$ is a pure number, of no physical dimensions. It has already been found that $v^2\rho\lambda/t$ possesses the same character, and no other such number can be formed independently. Between these two numbers exists a relation which is unknown by this simple reasoning, but can be expressed symbolically in the form

$$\phi(v^2\rho\lambda/t, \quad g\rho\lambda^2/t) = 0.$$

Many equivalent forms can be found. If t, ρ, λ are the predominant factors, and g is subsidiary, it is natural to select

$$v = \sqrt{\left(\frac{t}{\rho\lambda}\right)} \cdot f\left(\frac{g\rho\lambda^2}{t}\right),$$

where f is an undetermined function.

EXAMPLES.

1. A uniform chain of length l and mass M hangs vertically over a table, with the lower end at a height h above the surface. When the upper end is released and the chain falls, show that the pressure on the table when a length x has been arrested is $Mg(2h + 3x)/l$.

2. A smooth ball falling from rest at A on the inner surface of a sphere strikes an equal ball resting at the lowest point B, and causes it to rise to the point C on the sphere. If e is the coefficient of restitution between the balls, prove that the ratio of the chords BC, AB is $\frac{1}{2}(1 + e)$.

3. The masses of three spheres in a straight line are m_1, m, m_2, the coefficients of restitution between the successive pairs being e_1, e_2. If m_1 is projected with velocity V towards m, which after the impact hits m_2, prove that the greatest velocity which m_2 can acquire is

$$\frac{m_1(1 + e_1)(1 + e_2)}{(\sqrt{m_1} + \sqrt{m_2})^2} V,$$

and that this occurs when $m^2 = m_1 m_2$.

4. A perfectly elastic ball is thrown from a height h above the ground against a wall at a distance a, and returns to the thrower's hand after one impact on the ground. If (u, v) are the horizontal and vertical components of the velocity at projection, prove that

$$\frac{2h}{g} = \frac{a^2}{u^2} - \frac{v^2}{g^2},$$

and that $$(V^2 \cos 2\theta - 2hg)^2 = (V^2 + 2hg)^2 - 4a^2g^2,$$

if V is the velocity and θ the angle of projection.

5. A sphere is projected from a point on a smooth horizontal plane with the velocity V. Show that it will remain on the plane when it has reached the point at which it would have arrived in a single trajectory had it been projected in the same direction with the velocity $V(1-e)^{-\frac{1}{2}}$, e being the coefficient of restitution at the plane.

6. A smooth sphere of mass m_1 impinges on another of mass m_2, which is at rest, its line of approach making the angle θ with the line of centres at the moment of contact. If its path is deviated through the angle D by the impact, the coefficient of restitution being e, prove that

$$\tan D = \frac{m_2(1+e)}{2\{(m_1+m_2)(m_1-em_2)\}^{\frac{1}{2}}} \sin 2\phi,$$

where

$$\tan\phi = \left\{\frac{m_1+m_2}{m_1-em_2}\right\}^{\frac{1}{2}} \tan\theta,$$

provided $m_1 > em_2$. Find a similar formula for D when $m_1 < em_2$.

7. Equal balls n in number, each of mass m, are placed in a row, and the first is projected against the second with velocity v. When the impacts have reached the last ball, and before any pair have come in contact a second time, prove that the total loss of energy is

$$\frac{1+e}{3+e}\left[1-\left(\frac{1+e}{2}\right)^{2n-2}\right] mv^2,$$

e being the common coefficient of restitution.

8. A carriage on a switchback railway is made to loop the loop in a circle of radius R. What is the least height from which the carriage must start ?

9. A heavy uniform chain is hung over a smooth peg, with equal lengths a on either side. Masses m and $3m$ are then attached to the two ends. Show that when the chain is leaving the peg it is moving with the velocity $\sqrt{(ag)}$ whatever the weight of the chain may be.

10. A hammer of mass M falls from a height h on a hard inelastic pile of mass m, driving it down a distance a against a constant resistance R. Prove that the time during which the pile and the hammer are sinking together is

$$\frac{M+m}{M}\left(\frac{2}{gh}\right)^{\frac{1}{2}} a.$$

Calculate this time when 15 blows of a 2-ton hammer falling 5 feet are required to drive a pile weighing half-a-ton one foot into the ground.

11. A cylinder of mass m fits tightly in a corresponding hole in a block of mass M. Compare the penetration which will be produced by a given blow P, (*a*) when the block is held rigid and the blow is applied to the cylinder, (*b*) when the cylinder is fixed, and the blow is applied to the block.

12. A particle slides down the upper side of a smooth surface from rest at A, describing on the surface a curve which is in a vertical plane and concave downwards. If it leaves the surface at P, and C is the centre of curvature of the path at P, show that the depth of C below P is twice the depth of P below A.

13. Two equal masses connected by a light string are placed symmetrically on a smooth circle in a vertical plane, the string passing over the highest point. When they are disturbed the descending mass leaves the circle at the moment when the other reaches the top of the circle. Prove that the length of the string is one-fifth of the circumference of the circle.

14. A system consisting of n equal masses connected by $n-1$ equal light strings rests on a smooth platform. The first mass is dropped gently over and the other masses are set in vertical motion by $n-1$ successive jerks. Prove that just after the last jerk the kinetic energy of the system is $(2n-1)/3n$ of the work done by gravity up to that time.

Hence, or otherwise, determine the velocity when a length l of heavy chain originally coiled up on a smooth table has slipped over the edge.

15. A particle on a smooth table is connected by a light string passing through a smooth hole in the table to a particle of equal mass hanging freely. At a certain moment it is moving at right angles to the string with the velocity V_1 at the distance a from the hole. When it is next moving at right angles to the string its velocity is V_2 at the distance b from the hole. Express a and b in terms of V_1, V_2, and show that when the particle is again moving at right angles to the string the first conditions will be reproduced.

16. A clock pendulum, which is keeping correct time, has its amplitude increased from 3° to 4°. What effect is to be expected in the daily rate ?

17. A uniform heavy cable passes over a capstan from a coil on one side to another on the same level as the first. When the cable is passing at the steady speed v, and an equal length hangs from the capstan on either side, prove that the power of the agent which turns the capstan is mv^3, where m is the mass of the cable per unit length. Locate and account for the consumption of energy.

18. A steam pump delivers m lbs. of water per second at density ρ through an orifice of area A square feet at the height h feet above the intake. Prove that the effective horse-power of the pump is

$$\frac{mh}{550}\left(1+\frac{m^2}{2gh\,.\,\rho^2A^2}\right).$$

If the delivery is 1000 gallons per minute at a height of 300 feet, calculate the least horse-power required, and the diameter of the circular orifice which will involve an addition of 1 per cent.

19. A mass m performs simple harmonic vibrations of amplitude a and period T under the influence of a spring. Show that the spring develops power in a cycle with the period of $\frac{1}{2}T$.

If the mass is 4 lbs., the amplitude 2 ft., and the maximum power produced is 0·1 H.P., find the period of the vibrations.

20. A particle of mass m has its velocity increased from u to v in passing over the distance s. Prove that if the passage be made under constant power the time occupied will be less than if it be made with uniform acceleration.

21. Find the multiplier required to convert pressures expressed in tons' weight per square inch into kilogrammes per square cm.

22. A body travelling with the velocity v traverses a distance l in the time t, so that $v = Cl/t$, when the units of length, time and velocity are all independent. What is C, when the units are a foot, a second and a knot? A knot may be defined as one minute of arc per hour on a sphere of radius 3982 miles.

The *Mauretania* leaving port is observed to move through its own length of 785 feet in one minute. What is its speed in knots?

23. The mutual force of attraction between two particles of mass m, m' at distance r apart being $\gamma mm'/r^2$, what are the dimensions of γ?

Show that the acceleration of a particle in the presence of an infinite uniform thin sheet of matter, which depends on γ and the mass of the sheet per unit area, is independent of the distance of the particle from the sheet. Show also that the acceleration of a particle in the presence of an infinite uniform thin wire varies inversely as the distance from the wire.

24. The velocity of waves in a medium depends on the density of the medium and on some modulus which represents a force per unit of p dimensions in length. If the velocity $v \propto \lambda^n$, where λ is the wave-length, show that $n = \frac{1}{2}p - 1$.

25. The resistance R to a shot of given shape may be taken to depend on its diameter d, its velocity v, the velocity a of sound in air, the density ρ of air and its kinematic viscosity ν (of dimensions L^2T^{-1}). Show that the resistance can be represented in the form

$$R = \rho\, d^2 v^2 f\left(\frac{a}{v}, \frac{\nu}{dv}\right).$$

PART III

STATICS

CHAPTER VI

PARALLEL FORCES. CENTRES OF GRAVITY

148. The purpose of Statics is to study the conditions of the equilibrium of forces, when the bodies to which they are applied are at rest. The fundamental property, on which the relation of forces depends, is their vectorial character, and this follows from the definition of force as a producer of acceleration. This definition belongs naturally to dynamics, and for this reason the elements of dynamics precede the elements of statics. Strictly within the domain of statics, the law of combination of intersecting forces neither requires, nor is susceptible of, experimental proof. Experiments which purport to be made with this object at most prove that the forces supposed to be acting really are what they are supposed to be. If such an experiment appeared to fail it would not be right to infer that forces are not vectors, but merely that the forces assumed to be acting had been wrongly estimated.

In the limited part of dynamics already considered, forces have been supposed to act in effect on material points. All the forces acting at a point can be compounded into a single resultant, which can be calculated according to the parallelogram law. For equilibrium this resultant must vanish. Thus the statics of a single particle is extremely simple.

But the subject will now be extended to rigid bodies, and even connected systems, though generally with the limitation that the forces act in a single plane. One fact must be noted immediately. So long as the forces act at a single point they are defined completely by their amount and direction. But now the point of

application is also important. A force may be supposed to act on a rigid body at any point in its own line, but the line of action of a force is a necessary part of its specification. The force cannot be transferred to a parallel line without altering its action on the body. A force is therefore called a *localized vector*.

149. The following formal principle, or axiom, deserves to be stated explicitly. If two systems of forces, represented symbolically by S_1 and S_2, produce equilibrium when acting together on a rigid body, then the system derived from S_2 by reversing the forces and denoted by $-S_2$ will be the equivalent of S_1. In other words, since both the systems S_1 and $-S_2$ can be reduced to equilibrium by the addition of the system $+S_2$, either system acting alone will have the same effect and can be replaced by the other.

In particular, if S is a system of forces in equilibrium, any single force which is a member of S, reversed in direction, is the resultant of all the other forces of the system.

150. There is one simple system of four forces which clearly gives equilibrium, though the forces do not all pass through one point. It is the system represented in magnitude and position by the four sides of a parallelogram, and directed as in the figure. The

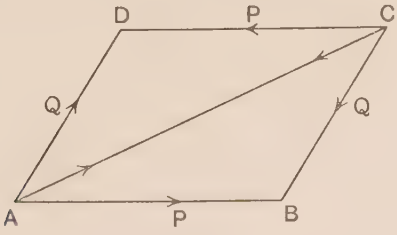

FIG. 74.

resultant of $AB=P$ and $AD=Q$ is $AC=R$, and the resultant of $CD=P$ and $CB=Q$ is the equal and opposite force $CA=R$. Since the system is equivalent to two equal and opposite forces acting on the same straight line it is in equilibrium.

Now this system consists of two pairs of equal, parallel and opposite forces $\pm P$, $\pm Q$.

151. *Def.* Two equal, parallel and opposite forces form what is called a *couple*. The separation, or perpendicular distance apart, of the two forces is called the *arm* of the couple.

Def. The *moment* of a force about a point is the product of the force into the perpendicular from the point on its line of action.

This definition needs to be supplemented by a convention as to the sign of the moment. In the figure the force is represented by

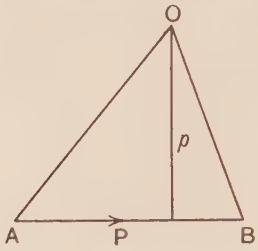

FIG. 75.

$AB=P$, and the perpendicular from O by p. The moment of P about O is represented by $p \,.\, P=p \,.\, AB=2$ triangle OAB.

But the moment has a sign, which is reversed when the direction of P is reversed. In the case represented, where the moment is usually taken to be positive, the force tends to produce a rotation about O in the anti-clockwise direction. When the force tends to produce a clockwise rotation the moment about O is negative. Similarly, the area of the triangle OAB must here be regarded as a directed area, being taken positively when it lies to the left of the forward direction of the force, and negatively when it lies to the right. With this convention the moments of a coplanar system of forces can be added algebraically.

The moment of a couple about a point is independent of the position of the point. For three distinct positions of the point O

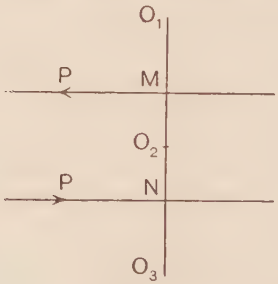

FIG. 76.

on the line MN perpendicular to the forces P, the moment is, according to the convention,

$$-O_1M \,.\, P+O_1N \,.\, P=MN \,.\, P,$$
$$O_2M \,.\, P+O_2N \,.\, P=MN \,.\, P,$$
$$O_3M \,.\, P-O_2N \,.\, P=MN \,.\, P,$$

or in all cases the product of the force by the arm of the couple. It is therefore represented graphically by the area of the parallelogram of which the opposite sides represent the two parallel forces.

152. The system of four forces in equilibrium, considered in § 150, consists of two couples. The moment of each couple is represented by the area of the parallelogram formed by the forces, but with opposite signs in the two cases. Hence the sum of the moments of this system is zero.

This result is more general than may appear at first sight, and the truth of the converse can be inferred. It must be remembered that a figure of the kind given is based on two independent scales, one representing linear distances and the other representing forces. In this case the two scales have been so adjusted that the equilibrium of the system is obvious. In deducing the converse the forces will now be represented on an arbitrary scale.

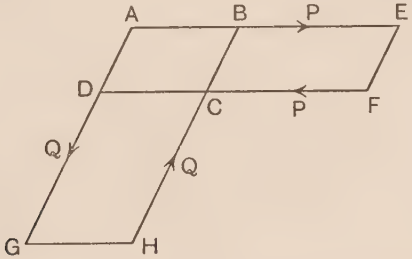

FIG. 77.

Let AE, FD on this scale represent the couple $\pm P$ (with negative moment) and AG, HB the couple $\pm Q$ (with positive moment).

If the moments of these couples are numerically equal the areas of the parallelograms $AEFD$, $ABHG$ are equal. Hence

$$\frac{AEFD}{ABCD}=\frac{ABHG}{ABCD},$$

and therefore

$$AE/AB=AG/AD.$$

By a change of scale in this ratio the couple $\pm P$ can be represented by AB, CD and the couple $\pm Q$ by AD, CB. The two couples together therefore form a system of forces in equilibrium.

By reversing the forces in one of the two couples it follows that two couples which have equal moments in the same sense are equivalent to one another. Therefore a couple can be moved in its own plane, and changed in any way, without altering its effect, provided its moment is preserved unaltered. In other words, the

effect of a couple in a given plane is completely specified by its moment, which is therefore its proper measure. A couple so measured is sometimes called a *torque*.

153. It is now possible to construct a system of three parallel forces in equilibrium. Let $ABCD$ be a parallelogram, and let

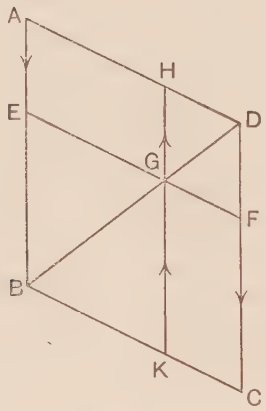

FIG. 78.

EF, HK be drawn parallel to the sides through any point G on the diagonal BD. Since parallelograms are bisected by their diagonals, the area $ABD=CBD$, $EBG=KBG$ and $HGD=GFD$. Hence the parallelograms $AEGH$, $CKGF$ are equal in area. Therefore, if forces are represented by AE, GH and KG, FC, they constitute two couples of which the moments are equal in magnitude and opposite in sense. Thus this system of three parallel forces AE, KH, FC is in equilibrium.

From the equilibrium of the system it immediately follows that any one of the three forces, reversed in direction, is the resultant of the other two parallel forces. Thus the resultant of the like forces $AE=P$, $FC=Q$ is the parallel force $HK=P+Q$. On the other hand, the resultant of unlike forces $KH=P$, $FC=Q$ is the parallel force $EA=P-Q$. When any two of the parallel forces are given in magnitude and position it is easy to complete the figure as drawn here, and so to construct the line of action of the resultant.

154. Let $AB=P$, $CD=Q$ be the given parallel forces. Construct the parallelogram $AEDF$ by drawing through A, D parallels to the forces and to BC. Let the diagonal EF meet BC in G. Through G draw a line parallel to the forces, meeting AF in H and ED in K.

Then $HK=R$ represents the resultant of P, Q in magnitude and position. Two figures are given, to represent the cases of two like

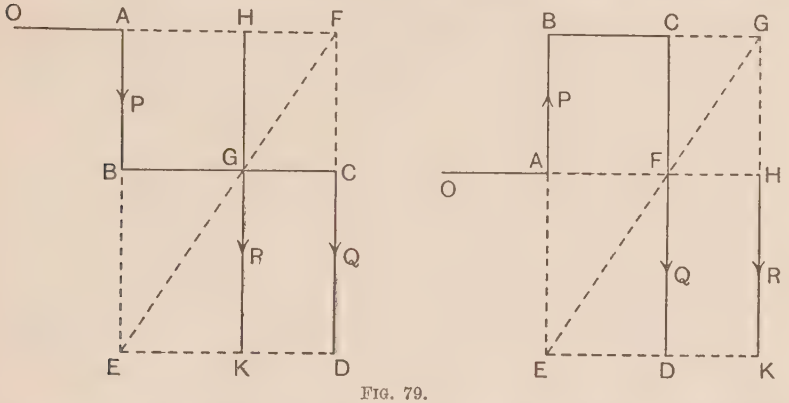

FIG. 79.

and two unlike forces, but it will be seen that the same construction applies equally to both, and reproduces the conditions represented in the earlier figure.

Since forces can be displaced along their lines of action, it is clear that, if O is any point in the plane, the line $OAHF$ may be taken at right angles to the forces, so making the parallelograms rectangles, without loss of generality.

In the first case (P and Q like forces) the rectangles $ABGH$, $GKDC$ are equal. Therefore

$$(OH-OA)P=(OF-OH)Q,$$
$$OH(P+Q)=OH\,.\,R=OA\,.\,P+OF\,.\,Q.$$

This equation defines the position of the point H or line HK, and expresses the fact that the moment of the resultant, $OH\,.\,R$, is equal to the sum of the moments of P and Q about O.

In the second case (P and Q unlike forces) the rectangles $BAFC$, $FDKH$ are equal. Therefore

$$(OF-OA)P=(OH-OF)R$$
$$=OH\,.\,R-OF(Q-P),$$

whence
$$OH\,.\,R=OF\,.\,Q-OA\,.\,P.$$

This again serves to define the position of the line HK, and states that the moment of the resultant is equal to the (algebraic) sum of the moments of P and Q about O.

The two cases of like and unlike parallel forces have been distinguished, in consequence of treating P and Q in both as positive

forces. By treating P as a negative force in the second case, both are reduced to a common rule, which is :

The resultant of two parallel forces is a parallel force, which is the algebraic sum of the two component forces. Its line of action is defined by the fact that its moment about any point is equal to the algebraic sum of the moments of the component forces about the same point.

155. By adding other parallel forces, one at a time, this rule can be generalized and stated in this form :

The resultant of a system of any number of parallel forces in a plane is a parallel force, of which the magnitude is the algebraic sum of the forces, and of which the moment about any point is equal to the algebraic sum of the moments of the forces. The resultant vanishes, and the system is in equilibrium, if the sum of the forces and the sum of the moments about a point vanish.

If the first of these conditions is satisfied, but not the second, the line of action of the last force considered does not coincide with that of the resultant of the rest, though in magnitude it is equal and opposite to that resultant. The system therefore reduces to a couple. From the equation for the moment of two unlike forces,

$$OH = \frac{OF \,.\, Q - OA \,.\, P}{R} = \frac{OF \,.\, Q - OA \,.\, P}{Q - P},$$

it is evident that, as P tends to equality with Q, the resultant tends to become very small, while OH increases without limit. The resultant of a couple may therefore be regarded as an infinitely small force acting at an infinite distance. Since this is not a practical conception, the couple, when existing alone, must be treated as irreducible.

Ex. The resultant of a couple, moment M, and the force F in the same plane is the same force F displaced parallel to itself through the distance M/F.

156. To understand the true nature of a couple, or torque, it is necessary to make a slight excursion into three dimensions.

Let $ABCDEFGH$ be a rectangular block, and let the parallel edges AB, CD, EF, GH represent equal forces P in the directions marked.

Let KL be drawn coinciding with that axis of the block which is parallel to these edges, and in length twice the length of the same edges. Clearly the resultant $2P$ of the forces AB, GH will be repre-

sented by KL, and the resultant $2P$ of the forces CD, EF will be represented by LK, which is equal and opposite to KL. Hence the

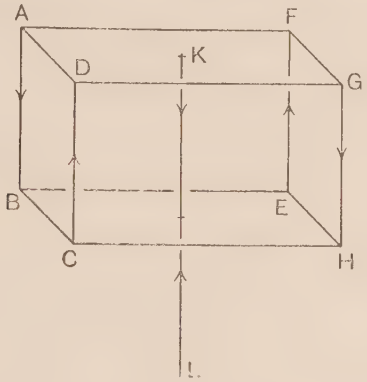

FIG. 80.

system of four parallel forces is in equilibrium. It then follows, on reversing two of these forces, that the couple AB, CD is equivalent to the couple FE, HG. Therefore a couple can be moved into a parallel plane without altering its effect. By combining this with a property already known, it becomes evident that the effect of a couple depends only on its moment and on the direction (but not the actual position) of its plane. Now the direction of a plane is equally defined by the direction of a line drawn perpendicular, or normal, to it.

157. The fact that the effect of a couple can be completely specified by a magnitude, its moment, associated with a directed line or axis, perpendicular to its plane, suggests that a couple is a vector, and this can now be proved. Let the planes of two couples intersect in AA'. The power of adjustment which a couple possesses

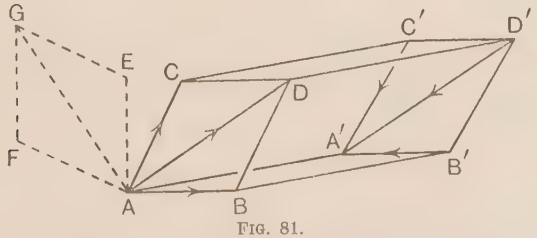

FIG. 81.

in its own plane makes it possible, without loss of generality, to represent the forces by AB, $B'A'$, AC, $C'A'$, all drawn at right angles

to AA'. The moments of these couples are $M_1 = AA' \,.\, AB$ and $M_2 = AA' \,.\, AC$. Complete the parallelopiped of which AA', AB, AC are intersecting edges, so forming a prism of which the end faces $ABDC$, $A'B'D'C'$ are parallelograms in planes at right angles to AA'.

The resultant of the forces AB, AC is AD, and the resultant of $B'A'$, $C'A'$ is $D'A'$, which is equal and parallel, but opposite, to AD. Both AD and $A'D'$ are at right angles to AA'. Hence the resultant of the two couples, whose moments are M_1 and M_2, is a couple, whose moment is $N = AA' \,.\, AD$.

Turn the parallelogram $ABDC$ in its own plane through a right angle, so that AB falls on AE, AD on AG and AC on AF. Let the lengths AE, AG, AF be such that AE represents $M_1 = AA' \,.\, AB$, AG represents $N = AA' \,.\, AD$ and AF represents $M_2 = AA' \,.\, AC$. Since these are in the same proportion as AB, AD, AC, the figure $AEGF$ is a parallelogram similar to $ABDC$. Also the line AE in the plane ABC is perpendicular to AA' and to AB, and so is normal to the plane $ABB'A'$. Similarly AF is normal to the plane $ACC'A'$, and AG is normal to the plane $ADD'A'$. Hence the two couples and their resultant are represented by AE, AF and AG both in magnitude and direction, and, since these lines are related according to the parallelogram law, couples are vectors. In magnitude a couple is measured by its moment, and in direction it is defined by its axis, or the normal to its plane.

Since a couple can be moved in its own plane, or into a parallel plane, without changing its effect, the actual position of its axis (apart from direction) is immaterial. Unlike a force, a couple is an unlocalized vector. Being a vector, any couple in space can be resolved along three given axes, and the three components define its action on a rigid body completely.

158. Let AB, CD represent two parallel forces P, Q acting through the points A, C, and let GH represent the parallel resultant $R = P + Q$. About the point C the moments of P and R are equal, so that, if the line CNM meets AB, GH at right angles,

$$CM \,.\, P = CN \,.\, R.$$

Through the fixed point A let the force AB be turned into any new position AE without change of magnitude. If CF and GK are drawn equal respectively to CD and GH and parallel to AE, they lie in the same plane $CGAE$. Let the line $CN'M'$ meet GK, AE

at right angles. Since $CM/CN=CA/CG=CM'/CN'$, it follows that

$$CM' \,.\, P=CN' \,.\, R,$$

or the moments of P and R about C in their new position are equal. Hence $R=P+Q$ acting along GK is the resultant of P, Q acting along AE, CF.

To express this result in its important aspect it may be stated thus. Being given a system of two parallel forces P, Q acting

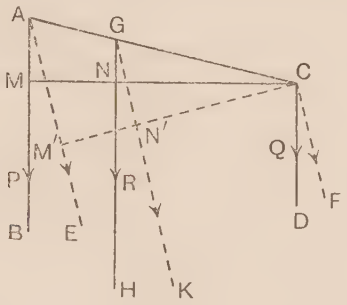

FIG. 82.

through given points A, C, the resultant $R=P+Q$ acts through a certain point G on AC, given by

$$CA \,.\, P=CG(P+Q),$$

and the position of G depends on the positions of A, C and on the magnitudes, but not the directions, of the parallel forces P, Q.

159. If a third parallel force S, acting through a definite point L, be added to the original system P, Q, the whole reduces first to $R=P+Q$ acting through G and S acting through L, and then to a single parallel resultant $R+S=P+Q+S$ acting through a definite centre G', the position of which is independent of the direction of the three parallel forces. This process can be repeated, step by step, as new parallel forces are added to the system. Hence, generally, a system of any number of parallel forces, each acting through an assigned point, has a parallel resultant, which in magnitude is equal to the sum of the forces, and which acts through a definite centre, the position of which is independent of the direction of the forces.

In order to determine the distance of this centre from a given plane, the forces of the system may be turned in the direction perpendicular to the plane. Let the forces P_1, P_2, P_3, ..., P_n act

through the points $A_1, A_2, A_3, \ldots, A_n$, the perpendicular distances of which from the given plane are $h_1, h_2, h_3, \ldots, h_n$. Let G_2 be the

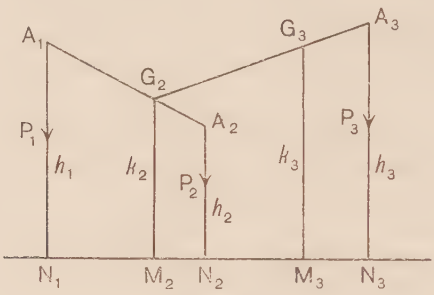

FIG. 83.

centre of the system P_1, P_2; G_3 of the system P_1, P_2, P_3; and so on, the distances of $G_2, G_3, \ldots, G_n$ from the plane being $k_2, k_3, \ldots, k_n$. Finally, let $N_1, N_2, \ldots$ be the feet of the perpendiculars from $A_1, A_2, \ldots$ and $M_2, M_3, \ldots$ the feet of the perpendiculars from $G_2, G_3, \ldots$ on the plane.

For the first partial system, P_1, P_2, by moments about A_2,

$$P_1 \cdot N_1N_2 = (P_1+P_2)M_2N_2.$$

But $N_1N_2/M_2N_2 = (h_1-h_2)/(k_2-h_2)$, and therefore

$$P_1(h_1-h_2) = (P_1+P_2)(k_2-h_2),$$

or

$$P_1h_1+P_2h_2 = (P_1+P_2)k_2.$$

Similarly, for the second partial system (P_1+P_2) through G_2 and P_3 through A_3,

$$(P_1+P_2)k_2+P_3h_3 = (P_1+P_2+P_3)k_3,$$

or

$$P_1h_1+P_1h_2+P_3h_3 = (P_1+P_2+P_3)k_3.$$

This process can be repeated and will clearly lead to the general result,

$$P_1h_1+P_2h_2+\ldots+P_nh_n = (P_1+P_2+\ldots+P_n)k_n,$$

and the distance of the centre of the whole system from the given plane can be written in the form

$$k_n = \frac{\Sigma Ph}{\Sigma P}.$$

The numerator of this fraction may be called the moment of the system of parallel forces with respect to the plane.

160. It is evident that the centre of any system of parallel forces with given points of application is a definite point, unless $\Sigma P = 0$, in which case the system, if not in equilibrium, reduces to a couple.

In considering the equilibrium of a given system of parallel forces, the intersections of their lines of action with any plane can be taken as the points of application. If the forces are then turned about these points into parallel positions in the plane, their resultant in the new position will be zero as in the old. The transformed system, which is one of parallel forces in a plane, is therefore in equilibrium. Hence the conditions of equilibrium for any system of parallel forces in space can be obtained by considering the conditions which apply to a plane system of parallel forces.

Ex. Parallel forces act through the points of a triangle, and are proportional to the opposite sides of the triangle. To find the centre of the system.

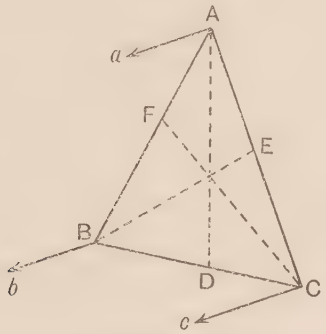

FIG. 84.

Let D in BC be the centre of the forces b at B and c at C. By moments about D, $b \,.\, BD = c \,.\, DC$, or

$$\frac{BD}{DC} = \frac{c}{b} = \frac{AB}{AC}.$$

Hence AD is the bisector of BAC. The centre of a at A and the resultant of b and c at D is a point on this bisector AD. Equally the centre of the three forces lies on BE, the bisector of ABC, and on CF, the bisector of ACB. The three bisectors must therefore meet in one point, which is in fact the centre of the inscribed circle of the triangle, and this point is the centre of the three parallel forces, irrespective of their direction in space.

161. *Ex.* A circular table, of weight W, is supported at the rim on four vertical legs, each of weight w, forming a square. A weight A is placed on the edge of the table, midway between two of the legs. Find the pressures of the legs on the ground, and the greatest value of A if the table is not to overturn.

[It is to be noticed that this problem, in the strict sense, is incapable of solution. This is obvious, because the removal of one leg would not necessarily affect equilibrium. Geometrically, the feet of three legs

determine contact with the ground, and the fourth is superfluous. In fact, the efficiency of this arrangement depends not only on good workmanship, but precisely on the fact that a table is not perfectly rigid. But here a symmetrical distribution of the forces is to be assumed.]

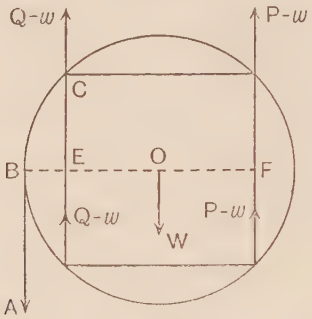

FIG. 85.

Let Q be the pressure exerted by the legs nearest to A on the ground, P that by the other pair of legs. The reactions being equal and opposite, when the forces are turned into the plane of the table, the forces on the table are as in the figure. The weight of the table, W, acts at its centre, and $OB = OC = OE\sqrt{2} = OF\sqrt{2}$. The first condition of equilibrium is clearly

$$A + W = 2(Q - w) + 2(P - w).$$

Also, by moments about O,

$$A\sqrt{2} = 2(Q - w) - 2(P - w).$$

Hence
$$Q = w + \tfrac{1}{4}W + \tfrac{1}{4}A(\sqrt{2} + 1),$$
$$P = w + \tfrac{1}{4}W - \tfrac{1}{4}A(\sqrt{2} - 1).$$

Now P cannot be negative, and when the table is on the point of overturning, $P = 0$. Therefore the greatest value of A is given by

$$\tfrac{1}{4}A(\sqrt{2} - 1) = w + \tfrac{1}{4}W,$$

or
$$A = (W + 4w)(\sqrt{2} + 1).$$

162. To consider the centre of a system of parallel forces further, let OYZ, OZX, OXY be three planes mutually at right angles. If the point P is the point of application of the force P, its position is defined by (x, y, z) the perpendiculars from P on these planes. If $(\bar{x}, \bar{y}, \bar{z})$ are the three corresponding perpendiculars from the point G, the centre of the system,

$$\bar{x} = \frac{\Sigma Px}{\Sigma P}, \quad \bar{y} = \frac{\Sigma Py}{\Sigma P}, \quad \bar{z} = \frac{\Sigma Pz}{\Sigma P},$$

and thus the position of G is completely determined.

It is to be noticed that the three coordinates remain quite distinct in these equations, so that $\bar{x}$, for example, depends in no

way on (y, z). It follows that in calculating $\bar{x}$, the point P can be transferred to P', its projection on the axis OX. The parallel

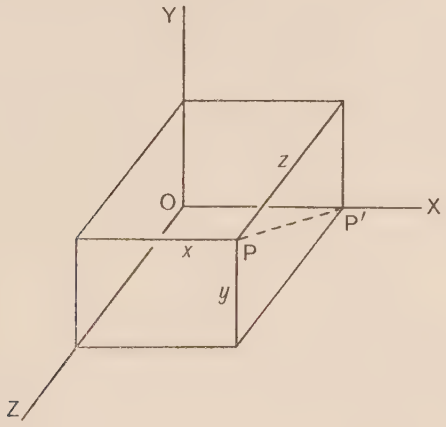

FIG. 86.

system can thus be changed into a system of parallel forces intersecting OX at right angles, and in this sense, since $OP' = x$, the moment ΣPx of the system with respect to the plane OYZ becomes the moment of the system about O. The other coordinates can be calculated by a similar process in turn.

Let the whole system be divided into two separate parts, $\Sigma_1 P$ and $\Sigma_2 P$, with corresponding moments $\Sigma_1 Px$ and $\Sigma_2 Px$ with respect to the plane YOZ. If $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$, $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$ refer to the centres of these partial systems, G_1 and G_2,

$$\bar{x}_1 \Sigma_1 P = \Sigma_1 Px, \quad \bar{x}_2 \Sigma_2 P = \Sigma_2 Px.$$

Hence
$$\bar{x} = \frac{\Sigma_1 Px + \Sigma_2 Px}{\Sigma_1 P + \Sigma_2 P} = \frac{\bar{x}_1 \Sigma_1 P + \bar{x}_2 \Sigma_2 P}{\Sigma_1 P + \Sigma_2 P},$$

with similar equations for $\bar{y}$, $\bar{z}$. If, again, from a complete system ΣP a part $\Sigma_1 P$ is removed, the centres being known in both cases, the centre of the remaining forces is given by

$$\bar{x}_2 = \frac{\Sigma_2 Px}{\Sigma_2 P} = \frac{\Sigma Px - \Sigma_1 Px}{\Sigma P - \Sigma_1 P} = \frac{\bar{x} \Sigma P - x_1 \Sigma_1 P}{\Sigma P - \Sigma_1 P},$$

with similar expressions for y_2, z_2. The three centres are in one straight line $G_1 G_2$, divided at G inversely as $\Sigma_1 P$ to $\Sigma_2 P$.

163. When a body is near the surface of the earth, each part of it, however small, possesses weight, which is the force of attraction exerted by the earth. The whole of these forces form a parallel

system, of which the resultant constitutes the weight of the body. This total weight is therefore the sum of the constituent weights, and acts on a rigid body as a single force through the centre of the parallel system. The points of application are fixed in the body, and the centre, in this case called the *centre of gravity* of the body, is also fixed in the body, its position being independent of the direction in which the weight is acting.

The mass of a constituent particle being m, its weight is $w=mg$, and since $\Sigma mgx/\Sigma mg=\Sigma mx/\Sigma m$, the distance of the centre of gravity of the body from three planes at right angles is given by

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}, \quad \bar{y}=\frac{\Sigma my}{\Sigma m}, \quad \bar{z}=\frac{\Sigma mz}{\Sigma m},$$

where $\Sigma m=M$, the mass of the body. In the first instance, in order to localize the points of application of the forces, the particles may be considered to be reduced to evanescent size, or the physical limit. But the formulae remain true, irrespective of the size of the constituent parts, if (x, y, z) are taken to refer to the centre of gravity of m. The position of the centre of gravity may be, as in the case of a cube of uniform density, obvious by symmetry.

The term "centre of gravity" is only appropriate when applied to a small body at the earth's surface, where the gravitational forces are sensibly parallel. It becomes absurd when applied, for example, to the solar system. The position of the point depends only on the distribution of mass, and not on the value of g. Hence a more appropriate term would be *centre of mass*, since the point will be found to possess a far wider significance than that appearing from its present derivation.

If the problem of finding the centre of gravity requires the body to be divided into an indefinite number of small elements, the summations become integrations. The element of mass being $\delta m=\rho\delta v$, where δv is the element of volume and ρ is the density, the equations take the form

$$\bar{x}=\frac{\int\rho x dv}{\int\rho dv}, \quad \bar{y}=\frac{\int\rho y dv}{\int\rho dv}, \quad \bar{z}=\frac{\int\rho z dv}{\int\rho dv},$$

where $\int\rho\, dv=M$ is the mass of the body, and the integrals in the numerators are taken over the whole volume of the body. If the density is uniform, ρ is constant and may be omitted from the

numerators and from the common denominator, which then becomes the total volume of the body.

164. Elementary cases will now be considered, in which the position of the centre of gravity is not obvious from symmetry, as in the circle or sphere.

To find the centre of gravity of a uniform triangle ABC, let D, E, F be the middle points of the sides. When these are joined

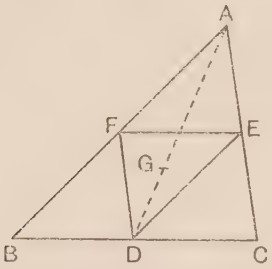

FIG. 87.

the triangle is divided into four equal similar triangles of mass $\frac{1}{4}M$ if M is the mass of ABC, and $AEDF$ is a parallelogram of mass $\frac{1}{2}M$, with its centre of gravity on the diagonal FE parallel to BC. Let p be the distance of A from the side BC and $\bar{x}$ the distance of the centre of gravity of the triangle ABC from the same side. The distance of the centres of gravity of FBD, EDC (on half the linear scale of ABC) from BC is clearly $\frac{1}{2}x$, and of $AFDE$ is $\frac{1}{2}p$. Hence

$$M\bar{x}=2\,.\,\tfrac{1}{4}M\,.\,\tfrac{1}{2}\bar{x}+\tfrac{1}{2}M\,.\,\tfrac{1}{2}p,$$

$$\tfrac{3}{4}x=\tfrac{1}{4}p,\quad \bar{x}=\tfrac{1}{3}p.$$

The same result would follow if a particle of mass $\frac{1}{3}M$ were placed at A and $\frac{2}{3}M$ in any way on BC. Similarly, the centre of gravity of ABC is at the same distance from AC as the centre of gravity of a particle $\frac{1}{3}M$ placed at B together with $\frac{2}{3}M$ placed on AC. Both these conditions are satisfied by placing particles of mass $\frac{1}{3}M$ at A, B and C. This system of three equal particles therefore has the same centre of gravity as the triangle.

But the two particles at B and C can be replaced by $\frac{2}{3}M$ at the middle point D, and the addition of $\frac{1}{3}M$ at A brings the centre of gravity of the whole to the point G on AD, where $AG=2GD$. As the centre of gravity of the triangle is a definite point, G is the common intersection of AD, BE, CF, and all these lines are trisected at G.

165. To deduce the centre of gravity of a uniform plane quadrilateral $ABCD$, let AC, BD meet in E. Let α be the mass of the

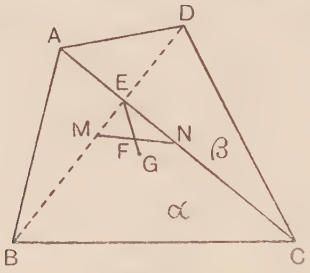

FIG. 88.

triangle ABC, and β the mass of the triangle ADC, so that $\alpha+\beta=M$, the total mass.

The triangle ABC may be replaced by masses $\frac{1}{3}\alpha$ at A, B, C, and the triangle ADC by masses $\frac{1}{3}\beta$ at A, D, C, or the whole quadrilateral by $\frac{1}{3}(\alpha+\beta)=\frac{1}{3}M$ at A, C, together with $\frac{1}{3}\alpha$ at B and $\frac{1}{3}\beta$ at D. This equivalent system of masses can again be replaced by $\frac{1}{3}M$ at the four points A, B, C, D, together with the negative masses $-\frac{1}{3}\beta$ at B and $-\frac{1}{3}\alpha$ at D. But $DE/EB=ADC/ABC=\beta/\alpha$, or

$$\alpha \,.\, DE=\beta \,.\, EB.$$

Hence $-\frac{1}{3}\beta$ at B and $-\frac{1}{3}\alpha$ at D can be replaced by $-\frac{1}{3}(\alpha+\beta)=-\frac{1}{3}M$ at E. It follows that the centre of gravity of the whole quadrilateral coincides with the centre of gravity of the system of particles consisting of $\frac{1}{3}M$ at the four corners A, B, C, D, together with the negative mass $-\frac{1}{3}M$ at the intersection E of the diagonals.

To construct the centre of gravity, the system of four equal particles first reduces to two, $\frac{2}{3}M$, at M, N, the middle points of the diagonals, and then to one, $\frac{1}{3}M$, at F, the middle point of MN. By combining this with $-\frac{1}{3}M$ at E, the centre of gravity is the point G on EF produced, where $FG=\frac{1}{3}EF$.

When two sides of the quadrilateral, $AB=a$, $CD=b$, are parallel, let E be the middle point of AB and, the diagonals intersecting in H, let EH meet CD in F. By similar triangles,

$$FC/AE=FH/HE=DF/EB,$$

or $FC/DF=AE/EB=1$. Hence F is the middle point of CD, and the system of particles equivalent to the quadrilateral reduces to $\frac{2}{3}M$ at E, F and $-\frac{1}{3}M$ at H. The last can be replaced by

$$-\tfrac{1}{3}Ma/(a+b)$$

at F and $-\frac{1}{3}Mb/(a+b)$ at E, since $FH/HE=FC/AE=b/a$. This reduces the equivalent system to

$$\tfrac{1}{3}M\,.\,\frac{2a+b}{a+b} \text{ at } E, \quad \tfrac{1}{3}M\,.\,\frac{2b+a}{a+b} \text{ at } F.$$

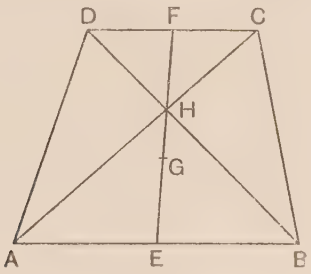

Fig. 89.

Hence the centre of gravity is the point G on EF, such that

$$\frac{EG}{GF}=\frac{2b+a}{2a+b}.$$

The case of the triangle is reproduced when $b=0$.

166. To find the centre of gravity of the uniform tetrahedron $ABCD$, let B_1, C_1, D_1 be the middle points of CD, DB, BC, and

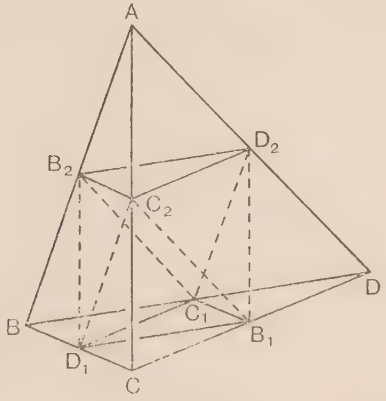

Fig. 90.

B_2, C_2, D_2 the middle points of AB, AC, AD. The tetrahedra $AB_2C_2D_2$, $B_2BD_1C_1$, $C_2D_1CB_1$, $D_2C_1B_1D$ are all equal and similar, and being on half the scale of $ABCD$ have each the mass $\frac{1}{8}M$, where M is the mass of $ABCD$. These four corners thus account for the mass $\frac{1}{2}M$, and the remaining solid, which is the octahedron

$$B_1C_1D_1B_2C_2D_2,$$

must also have the mass $\frac{1}{2}M$.

It is easy to see that the sections $C_1D_1C_2D_2$, $B_1D_1B_2D_2$, $B_1C_1B_2C_2$ of this solid are parallelograms, each dividing the octahedron into two equal, similar parts. The centre of gravity of the octahedron is therefore the common intersection of these three planes, or again, the common intersection of the three diagonals B_1B_2, C_1C_2, D_1D_2, and is thus situated in the plane midway between the planes $B_1C_1D_1$, $B_2C_2D_2$.

Let p be the height of A above the plane BCD, so that $\frac{1}{2}p$ is the distance between the parallel planes $B_2C_2D_2$ and BCD, and $\frac{1}{4}p$ is the height of the centre of gravity of the octahedron $B_1C_1D_1B_2C_2D_2$ above the plane BCD. Let z be the height of the centre of gravity of the whole tetrahedron $ABCD$ above the plane BCD, so that $\frac{1}{2}z$ is the height of the centres of gravity of the tetrahedra $B_2BD_1C_1$, $C_2D_1CB_1$, $D_2C_1B_1D$ and $\frac{1}{2}(p+z)$ is the height of the centre of gravity of $AB_2C_2D_2$. Then

$$M\bar{z}=3\,.\,\tfrac{1}{8}M\,.\,\tfrac{1}{2}\bar{z}+\tfrac{1}{8}M\,.\,\tfrac{1}{2}(p+\bar{z})+\tfrac{1}{2}M\,.\,\tfrac{1}{4}p,$$

$$\tfrac{3}{4}\bar{z}=\tfrac{3}{16}p,\quad \bar{z}=\tfrac{1}{4}p.$$

This shows that the height of the centre of gravity is the same as that of $\frac{1}{4}M$ placed at A, together with $\frac{3}{4}M$ in the plane BCD. Hence the system of four equal masses $\frac{1}{4}M$ at A, B, C, D will possess the same centre of gravity as the solid tetrahedron.

Now the three particles at B, C, D can be replaced by their combined mass $\frac{3}{4}M$ at G_1, the centre of gravity of the triangle BCD, and the addition of $\frac{1}{4}M$ at A will bring the centre of gravity to the point G on AG_1, such that $AG=3GG_1$. If G_2, G_3, G_4 are the centres of gravity of the other faces, the point G similarly lies on BG_2, CG_3, DG_4, which must therefore meet AG_1 in a common point, which divides each of these four lines in the ratio 3 : 1.

167. It is to be remarked that the principle by which one distribution of mass has been replaced by another possessing the same centre of gravity is special to the purpose, and the results are not in general to be employed with any other object. With this limitation of the meaning of an equivalent system it will be clear that the masses can be multiplied by any constant factor, and the scale of the system can be altered in any ratio, without disturbing the position of the centre of gravity.

Cases in which the centre of gravity can be found without dividing the body into small elements, and performing an infinite summation or integration, are comparatively rare. A straight rod,

with the density varying as a power of the distance from an external point on its length, will be taken as an illustration of the process usually necessary.

Let AB be the rod and O the external point, such that $OA=a$, $OB=b$. At a point P on the rod, where $OP=x$, the linear density

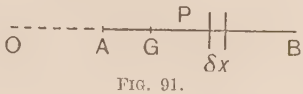

FIG. 91.

(or mass per unit length) is $\rho=\alpha x^p$, so that the mass of a small length δx of the rod at P is $\alpha x^p \,.\, \delta x$. The position of the centre of gravity is given by

$$OG=\bar{x}=\frac{\Sigma\alpha x^{p+1}\delta x}{\Sigma\alpha x^p\delta x}=\frac{\int_a^b x^{p+1}dx}{\int_a^b x^p dx}$$

$$=\left[\frac{x^{p+2}}{p+2}\right]_a^b \div \left[\frac{x^{p+1}}{p+1}\right]_a^b$$

$$=\frac{p+1}{p+2}\cdot\frac{b^{p+2}-a^{p+2}}{b^{p+1}-a^{p+1}}.$$

When O coincides with A, $a=0$ and

$$AG=\bar{x}=\frac{p+1}{p+2}\,b.$$

168. This result is very important because the problem of finding the centre of gravity can often be made to depend upon it.

Every material body has extension in three dimensions. But it is sometimes convenient to treat a body as a mere surface or shell when it has a small but uniform thickness, or even as a line when it has a small but uniform cross-section. In the first case, the thickness being t and an element of surface δs, the element of volume is $t \,.\, \delta s$, and the element of mass is $\rho t \,.\, \delta s=\sigma \,.\, \delta s$, where ρ is the density in the ordinary sense of mass per unit volume, and σ is the surface density or mass per unit area. Similarly in the second case, the area of the cross-section being a and an element of length δl, the element of volume is $a \,.\, \delta l$ and the element of mass $\rho a \,.\, \delta l=\lambda \,.\, \delta l$, where ρ is the volume density as before, and λ is the line density or mass per unit length. The dimensions of ρ, σ, λ are thus ML^{-3}, ML^{-2}, ML^{-1} respectively.

To find the centre of gravity of any solid cylinder of uniform density, with plane parallel ends, let G_1, G_2 be the centres of gravity of these ends.

The cylinder being constructed by drawing parallel lines in any direction through every point of a section, every parallel section is

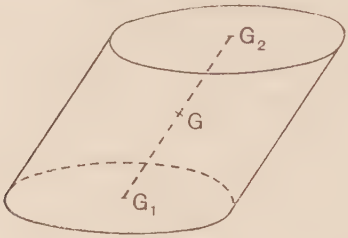

FIG. 92.

equal and similar. The cylinder can be divided into equal thin parallel sections, each with its centre of gravity on G_1G_2, and can be replaced by a thin rod G_1G_2 of uniform density. In this case $p=0$, and $G_1G=GG_2=\frac{1}{2}G_1G_2$, a result which is obvious by symmetry. A prism is a particular case.

The same result applies to a uniform hollow cylinder (or cylindrical surface) with open parallel ends. But care must be taken to interpret the meaning of the points G_1, G_2, which are the centres of gravity of thin sections of the cylinder. These are rings, in general of small but varying cross-section, and their centres of gravity cannot be identified with the result of treating them as uniform perimeters. This is possible only when the parallel end sections are normal to the generators of the cylinder, or when these sections have two axes of symmetry, the intersection of which defines the centre of gravity. In the solid cylinder a similar question does not arise, and in that case G_1, G_2 are the centres of gravity of the end sections treated as plane areas.

169. The plane triangle can be treated in a similar way. If D is the middle point of the side BC, AD bisects every line drawn parallel

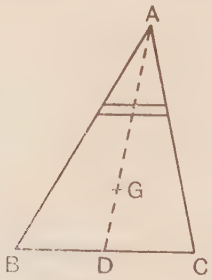

FIG. 93.

to BC. If the triangle is divided into thin strips of equal width parallel to BC, the centre of gravity of each lies on AD. The length, area and mass of each thin strip is proportional to its distance from A, and the triangle can thus be replaced by a distribution of mass along AD, varying as the distance from A. This reduces the triangle to a thin rod AD with $\rho=1$, and therefore if G be the centre of gravity of the triangle $AG=\frac{2}{3}AD$.

A similar form of reasoning and result will apply to the hollow cone, with plane open end. For sections parallel to the base are similar and have perimeters proportional to their distances from the vertex V. Hence, when the cone is divided into thin parallel sections of equal thickness, their centres of gravity lie on VG_1, where G_1 is the centre of gravity of the lowest section.

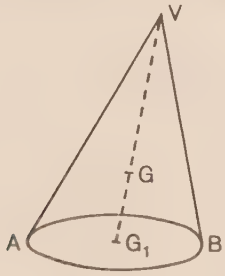

FIG. 94.

The cone can therefore be replaced by a thin rod VG_1, with density varying as the distance from $V(\rho=1)$, and its centre of gravity is the point G on VG_1, such that $VG=\frac{2}{3}VG_1$. As in the case of the cylinder, the point G_1 is not to be identified with the centre of gravity of the perimeter of the base, but can only be so taken when the base has two axes of symmetry and the vertex of the cone lies vertically above their intersection. Even in the case of the oblique hollow cone on a circular base, G_1 is not the centre of the base, and all that is to be inferred is that G lies in the plane parallel to the base at a height which is one-third of the height of the cone.

170. The tetrahedron is only a particular case of the uniform solid cone on any plane base. Sections parallel to the base are similar, and their areas are proportional to the square of the distance from the vertex V. Hence, if the cone is divided into thin sections of equal thickness parallel to the base, and G_1 is the centre of gravity of the lowest section, which can be treated as a plane area for the purpose of finding G_1, the whole can be replaced by a distribution

of mass along VG_1 varying as the square of the distance from V. This reduces the cone to a thin rod VG_1, the density along which is defined by $p=2$, and hence $VG=\frac{3}{4}VG_1$, G being the centre of gravity of the rod or cone.

If the cone be divided into two parts by a plane parallel to the base, and the top, which is a similar cone, be removed, the centre of gravity of the remaining frustum can be found in the usual way. Or more directly, if G_2 is the centre of gravity of the upper plane surface, the frustum can be reduced as before to a thin rod G_2G_1, of which the density varies as the square of the distance from the external point V in the same line. Hence, if $VG_2=a$, $VG_1=b$, and G is the centre of gravity of the frustum,

$$VG=\tfrac{3}{4}\cdot\frac{b^4-a^4}{b^3-a^3}.$$

Thus G divides the line G_1G_2 in the ratio

$$\frac{G_2G}{GG_1}=\frac{VG-a}{b-VG}=\frac{3b^4-4ab^3+a^4}{b^4-4a^3b+3a^4}$$
$$=\frac{(b-a)^2(a^2+2ab+3b^2)}{(b-a)^2(b^2+2ab+3a^2)}$$
$$=\frac{A+2\sqrt{(AB)}+3B}{B+2\sqrt{(AB)}+3A},$$

where A is the area of the top and B is the area of the base, for $A/a^2=B/b^2$. The complete cone ($A=0$) and the cylinder ($A=B$) are particular cases.

171. When circular or spherical boundaries are involved, the problem of finding the centre of gravity is in general less simple,

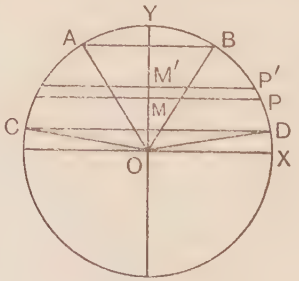

FIG. 95.

but the necessary integration can often be effected by first principles.

To find the centre of gravity of a strip of a circle enclosed between parallel chords, AB, CD, let OX be the parallel radius and OY the

perpendicular radius. Let R be the radius and $\sigma=1$ the surface density. The strip is divided into n narrow strips by chords parallel to OX, through $P(x_r, y_r)$, $P'(x_{r+1}, y_{r+1})$, ..., the range being limited by $D(x_0, y_0)$ and $B(x_n, y_n)$. The moment of the strip PP' with respect to OX is

$$y \cdot 2x(y_{r+1}-y_r)=x(y^2{}_{r+1}-y_r{}^2),$$

where $x_{r+1}<x<x_r$, $2x$ being the length of the strip, and $y_{r+1}>y>y_r$, y being the average distance of the strip from OX, for which

$$\tfrac{1}{2}(y_{r+1}+y_r)$$

is an intermediate value. But

$$R^2=x^2{}_{r+1}+y^2{}_{r+1}=x_r{}^2+y_r{}^2.$$

Hence the moment of the elementary strip is

$$\begin{aligned} x(x_r{}^2-x_{r+1}{}^2)&=x(x_r+x_{r+1}) \cdot (x_r-x_{r+1}) \\ &=2x^2(x_r-x_{r+1}), \end{aligned}$$

where $x_{r+1}<x<x_r$, although the meaning of x has been slightly changed.

The moment about OX of the whole strip $ABDC$ is therefore

$$\begin{aligned} 2\Sigma x^2(x_r-x_{r+1})&=2\int_{x_n}^{x_0} x^2 dx \\ &=\tfrac{2}{3}\Big[x^3\Big]_{x_n}^{x_0}=\tfrac{2}{3}(x_0{}^3-x_n{}^3). \end{aligned}$$

Let $BOY=\beta$, $DOY=\alpha$, so that $x_0=R\sin\alpha$, $x_n=R\sin\beta$, and the moment becomes $\frac{2}{3}R^3(\sin^3\alpha-\sin^3\beta)$. But the area or mass (with $\sigma=1$) of the strip is

$$2ODY-COD-(2OBY-AOB)=R^2(\alpha-\tfrac{1}{2}\sin 2\alpha-\beta+\tfrac{1}{2}\sin 2\beta).$$

Hence, if the point G, situated on OY by symmetry, is the centre of gravity of the strip,

$$OG=\tfrac{2}{3}R \cdot \frac{\sin^3\alpha-\sin^3\beta}{2(\alpha-\beta)-(\sin 2\alpha-\sin 2\beta)}.$$

The strip becomes a segment of the circle when $\beta=0$, and in this case the moment becomes $\frac{2}{3}R^3\sin^3\alpha$, the mass $\frac{1}{2}R^2(2\alpha-\sin 2\alpha)$, and

$$OG=\tfrac{4}{3}R \cdot \frac{\sin^3\alpha}{2\alpha-\sin 2\alpha}.$$

When $\alpha=\frac{1}{2}\pi$ the segment becomes a semi-circle, and in this particular case

$$OG=4R/3\pi.$$

172. The centre of gravity of the sector $OCYD$ of a circle is easily deduced. Let G_1 be the centre of gravity of the triangle

COD. Since $R\cos\alpha$ is the length of the perpendicular from O on CD, $OG_1=\frac{2}{3}R\cos\alpha$. The mass ($\sigma=1$) of the triangle is $\frac{1}{2}R^2\sin 2\alpha$, and therefore the moment about OX is $\frac{1}{3}R^3\sin 2\alpha\cos\alpha$. As above, the mass of the segment CYD is $\frac{1}{2}R^2(2\alpha-\sin 2\alpha)$ and the corresponding moment is $\frac{2}{3}R^3\sin^3\alpha$. Hence for the sector $OCYD$

$$OG=\frac{\frac{2}{3}R^3\sin\alpha\cos^2\alpha+\frac{2}{3}R^3\sin^3\alpha}{\frac{1}{2}R^2\sin 2\alpha+\frac{1}{2}R^2(2\alpha-\sin 2\alpha)}$$

$$=\frac{2}{3}R\,.\,\frac{\sin\alpha}{\alpha}$$

with the particular result for the semi-circle ($\alpha=\frac{1}{2}\pi$), $OG=4R/3\pi$ as before.

Let the same sector $OCYD$ be divided by a concentric circle of radius R' into the sector $OC'D'$ and the area $CC'D'D$. The mass of

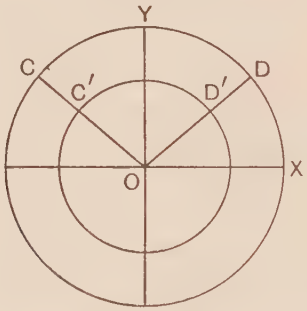

FIG. 96.

the sector OCD is $R^2\alpha$ and its moment (represented by the numerator of the above expression for OG) about OX is $\frac{2}{3}R^3\sin\alpha$. Similarly the mass of the sector $OC'D'$ is $R'^2\alpha$ and its moment $\frac{2}{3}R'^3\sin\alpha$.

Hence, if G is the centre of gravity of the area $CC'D'D$,

$$OG=\frac{\frac{2}{3}R^3\sin\alpha-\frac{2}{3}R'^3\sin\alpha}{R^2\alpha-R'^2\alpha}$$

$$=\frac{2\sin\alpha}{3\alpha}\,.\,\frac{R^2+RR'+R'^2}{R+R'}\,.$$

This result is true for any value of R', and therefore, when R' is made to approach R, the area degenerates into the arc CD of uniform line density, and in this case

$$OG=R\,.\,\frac{\sin\alpha}{\alpha},$$

where 2α is the angle subtended by the arc at the centre. In the particular case of a semi-circular arc, $\alpha=\frac{1}{2}\pi$ and $OG=2R/\pi$.

173. To find the centre of gravity of a section $ABCD$ of a uniform sphere, radius R, contained between parallel planes AB, CD, let OY

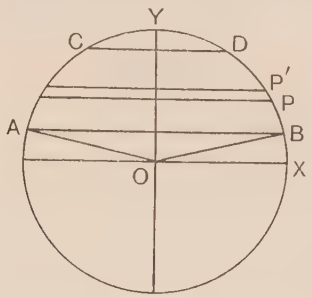

FIG. 97.

be the radius perpendicular to the planes and OX a radius at right angles to OY. If the terminal planes make intercepts y_0, y_n on OY, the interval can be divided at distances such as y_r, y_{r+1} from O, and the whole section into n parts by parallel planes.

The radii of the corresponding plane sections are x_r, x_{r+1} where

$$R^2 = x_r^2 + y_r^2 = x_{r+1}^2 + y_{r+1}^2.$$

If the density $\rho = 1$, the mass of an elementary section is

$$\pi x^2 (y_{r+1} - y_r) = \pi (R^2 - y^2)(y_{r+1} - y_r),$$

where $x_r > x > x_{r+1}$ and $y_r < y < y_{r+1}$, and the moment of this section about OX (or, more strictly, with respect to the plane at right angles to OY) is

$$\pi x^2 (y_{r+1} - y_r) y = \pi y (R^2 - y^2)(y_{r+1} - y_r).$$

Hence, for the whole section from y_0 to y_n the mass is

$$\Sigma \pi (R^2 - y^2)(y_{r+1} - y_r) = \pi R^2 \int_{y_0}^{y_n} dy - \pi \int_{y_0}^{y_n} y^2 \, dy$$
$$= \pi R^2 (y_n - y_0) - \tfrac{1}{3}\pi (y_n^3 - y_0^3),$$

and the moment is

$$\Sigma \pi y (R^2 - y^2)(y_{r+1} - y_r) = \pi R^2 \int_{y_0}^{y_n} y \, dy - \pi \int_{y_0}^{y_n} y^3 \, dy$$
$$= \tfrac{1}{2}\pi R^2 (y_n^2 - y_0^2) - \tfrac{1}{4}\pi (y_n^4 - y_0^4).$$

The centre of gravity by symmetry is a point G on OY, and is thus given by

$$OG = \frac{\tfrac{1}{2}\pi R^2 (y_n^2 - y_0^2) - \tfrac{1}{4}\pi (y_n^4 - y_0^4)}{\pi R^2 (y_n - y_0) - \tfrac{1}{3}\pi (y_n^3 - y_0^3)}$$
$$= \tfrac{3}{4} \cdot \frac{(y_n + y_0)(2R^2 - y_n^2 - y_0^2)}{3R^2 - (y_n^2 + y_0 y_n + y_0^2)}.$$

174. The section becomes a segment of the sphere when $y_n = R$, in which case

$$OG = \tfrac{3}{4} \cdot \frac{(R+y_0)(R^2-y_0^2)}{2R^2 - Ry_0 - y_0^2}$$
$$= \tfrac{3}{4} \cdot \frac{(R+y_0)^2}{2R+y_0}.$$

In the particular case of a solid hemisphere $y_0 = 0$ and

$$OG = \tfrac{3}{8}R.$$

If to the section $ABCD$ the cone AOB be added, the solid $AOBDC$ is obtained. For the cone the mass is $\frac{1}{3}\pi y_0 x_0^2 = \frac{1}{3}\pi y_0(R^2-y_0^2)$ and the height of the centre of gravity above O being $\frac{3}{4}y_0$, the moment about OX is $\frac{1}{4}\pi y_0^2(R^2-y_0^2)$.

Hence for this solid of revolution,

$$OG = \frac{\frac{1}{2}\pi R^2(y_n^2-y_0^2) - \frac{1}{4}\pi(y_n^4-y_0^4) + \frac{1}{4}\pi y_0^2(R^2-y_0^2)}{\pi R^2(y_n-y_0) - \frac{1}{3}\pi(y_n^3-y_0^3) + \frac{1}{3}\pi y_0(R^2-y_0^2)}$$
$$= \tfrac{3}{4} \cdot \frac{2R^2y_n^2 - R^2y_0^2 - y_n^4}{3R^2y_n - 2R^2y_0 - y_n^3}.$$

When $y_n = R$ the figure becomes a sector of the sphere, for which

$$OG = \tfrac{3}{4} \cdot \frac{R^2(R^2-y_0^2)}{2R^2(R-y_0)} = \tfrac{3}{8}(R+y_0),$$

and in the particular case of a solid hemisphere, $OG = \frac{3}{8}R$ as before.

175. Hollow spherical shells can be treated by special methods. In a central section of the sphere, of radius R, let OX, OY be two perpendicular radii, and XT parallel to OY the tangent at X. Taking

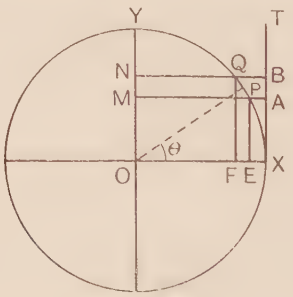

FIG. 98.

two close points P, Q on this section of the sphere, let A, B be the feet of the perpendiculars from P, Q on XT, M, N the feet of the perpendiculars on OY, and E, F the feet of the perpendiculars on OX. If θ is the angle between OX and the radius to the middle

point of PQ, this is also the angle between the chord PQ and PE or QF. Hence $AB=MN=PQ\cos\theta$ and $FE=PQ\sin\theta$. But $x=R\cos\theta$, $y=R\sin\theta$ are the perpendicular distances from OY, OX of the middle point of the arc PQ. With this meaning of x, y, therefore, $AB\,.\,R=PQ\,.\,x$ and $FE\,.\,R=PQ\,.\,y$.

Let the figure be now rotated about OY through an angle α. In this rotation PQ will generate an area $PQQ'P'$ on the sphere, AB will generate an area $ABB'A'$ on the enveloping cylinder generated by XT, and FE will generate an area $FEE'F'$ on the plane OX at right angles to OY. The cylindrical area $ABB'A'$ has the length $R\alpha$, the constant width AB and the area

$$AB\,.\,R\alpha=PQ\,.\,x\alpha.$$

The plane annular area $FEE'F'$ has the area

$$\tfrac{1}{2}\alpha(OE^2-OF^2)=\tfrac{1}{2}\alpha(OE+OF)EF=PQ\,.\,x'y\alpha/R,$$

where x' is the perpendicular distance from OY of the middle point of the chord PQ.

The middle point of PQ traces an arc of length $x\alpha$, and the area $PQQ'P'$ is therefore $PQ\,.\,x\alpha=\delta s$. This is the same as the area $ABB'A'$ on the enveloping cylinder. The areas, with the surface density $\sigma=1$, represent masses, and these elementary masses on the sphere and the enveloping cylinder are not only equal, but being situated at equal distances from the plane OX also have equal moments with respect to this plane at right angles to OY. Now any finite area on the sphere can be projected on the enveloping cylinder by lines drawn through OY parallel to the plane OX. When this is done, since the masses and moments of corresponding elements are equal, the centres of gravity of the spherical area and of its projection on the enveloping cylinder must lie at equal distances from the plane OX.

Thus, if two parallel planes intersect a sphere and the enveloping cylinder which touches the sphere along a third parallel plane, they include equal areas on both surfaces, and the centre of gravity of the spherical zone lies midway between the two parallel planes. In particular, the centre of gravity of a hemispherical shell lies at a distance from the centre equal to half the radius.

176. Again, if δA is the area $FEE'F'$, which is the orthogonal projection of δs on the plane OX,

$$R\,.\,\delta A=PQ\,.\,x'y\alpha=x'y\,.\,\delta s/x=y'\,.\,\delta s,$$

where y' is the perpendicular distance from OX of the middle point of the chord PQ, since $y'/x' = y/x$. Hence if A is the orthogonal projection of a finite area s on the sphere,

$$R \,.\, A = R \,.\, \Sigma \delta A = \Sigma y' \,.\, \delta s = y \,.\, s,$$

where y is the perpendicular distance from the plane OX of the centre of gravity of the area s. Thus $\bar{y} = R \,.\, A/s$.

Ex. In the case of a uniform shell in the form of an octant of a sphere, $s = 4\pi R^2/8$, $A = \pi R^2/4$ and $\bar{y} = \frac{1}{2}R$. Since this applies to each of the three planes formed by radii to the corners of the octant, the centre of gravity of the octant coincides with the centre of the cube of which the three radii are sides.

177. Similar methods have an application to surfaces of revolution. Let P, Q be two close points on a plane curve, PM, QN their

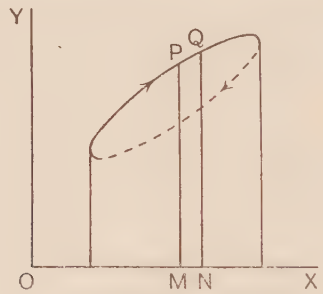

FIG. 99.

ordinates on the axis OX. If uniform density 1 be attributed to the curve, the arc $PQ = \delta s$ represents the mass of an element, y is the distance of its centre of gravity from OX, where $PM < y < QN$, and $\delta\mu = y \,.\, \delta s$ is the corresponding element of the moment of the curve about OX.

Let the figure be rotated about OX through an angle α, so that the curve generates part of a surface of revolution. The length of the arc traced by any point on PQ is $y\alpha$, where again $PM < y < QN$. The element of surface generated by $PQ = \delta s$ is therefore

$$\delta s \,.\, y\alpha = \alpha \,.\, \delta\mu.$$

Hence the surface generated by a finite length s of the curve is

$$\alpha\Sigma \,.\, \delta\mu = \alpha\Sigma y \,.\, \delta s = \alpha y \,.\, s,$$

where y is the ordinate of the centre of gravity of the curve, treated as uniformly loaded, and αy is the length of the circular arc traced by this point.

In the same rotation the area $PMNQ$ traces an element of the volume of the surface. The element of area is $y \,.\, MN = y \,.\, \delta x$, the ordinate of its centre of gravity is $\frac{1}{2}y$, and its moment about OX is $\delta\mu' = \frac{1}{2}y^2\delta x$. The area of the sector generated by the rotation of the ordinate of a point on PQ is $\frac{1}{2}y^2\alpha$, and the volume generated by the rotating element of area is $\frac{1}{2}y^2\alpha \,.\, \delta x$, where as before

$$PM < y < QN.$$

Hence the volume generated by a finite length of the curve is

$$\alpha\Sigma\tfrac{1}{2}y^2 \,.\, \delta x = \alpha\Sigma \,.\, \delta\mu' = \alpha\bar{y}' \,.\, A,$$

where A is the area of the curve and $\bar{y}'$ is the ordinate of its centre of gravity, so that $\alpha y'$ is the length of the circular arc traced by this point.

The area A is here bounded by parallel ordinates. If it be, on the contrary, a closed curve not intersecting the axis OX, it becomes the difference of two areas lying between bounding ordinates, so that $A = A_2 - A_1$, $\mu' = \mu_2' - \mu_1'$. The volume generated by the closed curve is now the difference of two volumes represented by $\alpha(\mu_2' - \mu_1') = \alpha y' \,.\, (A_2 - A_1) = \alpha y' \,.\, A$, where y' is still the centre of gravity of the closed area A. Similarly, in the case of surfaces, it is the sums of two curves and two moments which are involved, but the result remains as before, when $\bar{y}$ is taken to refer to the centre of gravity of the total perimeter. These theorems were given by Pappus (*circa* 300 A.D.).

178. *Ex.* Deduce the centre of gravity of a circular arc.

If the radius is R, and the angle 2α, the length of the chord is $2R\sin\alpha$. Let the complete circle be rotated about the diameter parallel to the chord, so generating a sphere. The area generated by the arc is equal to the area generated by a length $2R\sin\alpha$ on the tangent to the circle at the middle point of the arc, and is $2\pi R \,.\, 2R\sin\alpha$. The length of the arc is $s = 2R\alpha$. Hence if $\bar{y}$ is the distance of the centre of gravity of the arc from its centre,

$$2\pi R \,.\, 2R\sin\alpha = 2\pi\bar{y} \,.\, 2R\alpha,$$

or

$$\bar{y} = R\sin\alpha/\alpha.$$

Ex. Find the volume and area of an anchor ring or tore, generated by the rotation of a circle of radius R about a line in its plane, the distance from which of the centre of the circle is $a\,(>R)$.

Since a is the distance of the centre of gravity of the area from the axis of rotation, the volume is

$$2\pi a \,.\, \pi R^2 = 2\pi^2 a R^2.$$

Since a is also the distance of the centre of gravity of the perimeter from the axis, the area of the surface is $2\pi a \,.\, 2\pi R = 4\pi^2 a R$.

179. If G is the centre of gravity $(\bar{x}, \bar{y})$ of any number of masses m_1 at $A_1(x_1, y_1)$, m_2 at $A_2(x_2, y_2)$, ..., in a plane, and $M = \Sigma m$,

$$M\bar{x} = \Sigma mx, \quad M\bar{y} = \Sigma my.$$

Hence, if G coincides with the origin O, $\bar{x} = \bar{y} = 0$ and

$$\Sigma mx = \Sigma my = 0.$$

Let P be any point (h, k). Then, when the axes are drawn through G,

$$\begin{aligned}\Sigma m \,.\, PA^2 &= \Sigma m\{(h-x)^2 + (k-y)^2\} \\ &= \Sigma m \,.\, (h^2+k^2) - 2h \,.\, \Sigma mx - 2k \,.\, \Sigma my + \Sigma m(x^2+y^2) \\ &= M(h^2+k^2) + \Sigma m(x^2+y^2) \\ &= M \,.\, PG^2 + \Sigma m \,.\, GA^2.\end{aligned}$$

Hence $\Sigma m \,.\, PA^2$ is least when $PG = 0$, or P coincides with the centre of gravity of the system of masses. The result is equally true when the masses are not in a plane, and it is easy to extend the proof by the addition of a third coordinate.

In the next place, let P coincide in succession with $A_1, A_2, \ldots$. When it coincides with A_i,

$$m_i \sum_j m_j \,.\, A_iA_j^2 = m_i M \,.\, A_iG^2 + m_i \sum_j m_j \,.\, GA_j^2.$$

Hence, by summing all the equations of this type,

$$\sum_i m_i \sum_j m_j \,.\, A_iA_j^2 = M \sum_i m_i \,.\, A_iG^2 + \sum_i m_i \,.\, \sum_j m_j \,.\, GA_j^2,$$

or

$$2\Sigma m_i m_j \,.\, A_iA_j^2 = 2M\Sigma m_i \,.\, A_iG^2,$$

for the two complete sums on the right are identical, while on the left the term actually written down occurs in two ways, according as m_i is taken from the first or the second sum; factors of the form m_i^2 do not occur because they would be associated with $A_iA_i^2 = 0$. Thus, to combine the two results,

$$\begin{aligned}\Sigma m_i m_j \,.\, A_iA_j^2 &= M \,.\, \Sigma m \,.\, GA^2 \\ &= M \,.\, \Sigma m \,.\, PA^2 - M^2 \,.\, PG^2,\end{aligned}$$

the sum on the left being extended to every pair of masses.

Ex. The centre of gravity G of a tetrahedron coincides with that of four equal masses at the corners. In this case,

$$\Sigma A_iA_j^2 = 4\Sigma GA^2 = 4\Sigma PA^2 - 16PG^2.$$

Hence, if P is the centre of the circumscribing sphere, radius R,

$$PG^2 = R^2 - \tfrac{1}{16}\Sigma a^2,$$

where Σa^2 is the sum of the squares of the six edges.

EXAMPLES.

1. Prove the following construction for the resultant of two parallel forces AP, BQ: Draw BR equal and opposite to BQ, and let PR meet AB in E. Let M be the middle point of AB, and F the point of AB such that $ME = MF$. Through A and P draw lines respectively parallel to PR and AB. These lines intercept a length representing the resultant of AP, BQ on the line drawn parallel to them through F.

2. Forces are represented in magnitude and position by BA, AC. Show that they are equivalent to the force BC, together with a couple of which the moment is represented by twice the area ACB.

3. A system of forces is represented by the sides of a plane polygon taken in order. Show that they are equivalent to a couple, whose moment is represented by twice the area of the polygon.

4. On one line three points A, B, C are taken such that $AB = BC$, and on another line three points D, E, F such that $DE = EF$. Prove that the forces represented in magnitude and position by AD, BF, CE, DB, EA, FC form a system in equilibrium.

5. Couples act in the planes of the four faces of a tetrahedron. Their moments are proportional to the areas of the faces, and they have the same sense with respect to the interior of the tetrahedron. Show that they are in equilibrium.

6. If G is the centre of gravity of particles of mass $m_1, m_2, \ldots, m_n$ situated at the points $P_1, P_2, \ldots, P_n$, prove that $\Sigma m_r \,.\, OP_r$, the sum of the vectors OP_r from any point O, each multiplied by the corresponding mass m_r, is $OG \,.\, \Sigma m_r$.

7. Four forces whose lines of action are parallel act on a rigid body and keep it in equilibrium. Any plane intersects their lines of action in the points A_1, A_2, A_3, A_4. Prove that the magnitudes of the forces are proportional to the areas of the triangles $A_2A_3A_4$, etc.

8. The resultant of like parallel forces acting at the vertices of a triangle ABC passes through the centre of the circumscribing circle. Prove that the forces are proportional to $\sin 2A$, $\sin 2B$, $\sin 2C$.

9. The resultant of like parallel forces acting at the vertices of a triangle ABC passes through the orthocentre. Prove that the forces are proportional to $\tan A$, $\tan B$, $\tan C$.

10. On a triangular area ABC are placed parallel wires of uniform gauge terminated by AB, AC. One is placed on BC and n others divide AB, AC into $n + 1$ equal parts. If D is the middle point of BC and G is the centre of gravity of the wires, prove that $AG/GD = 2 + 3/n$.

11. The three sides of a right-angled triangle are a (the hypotenuse), b and c, and on these sides squares are drawn. Find the distances of the centre of gravity of the figure formed by the three squares from the three sides of the triangle.

12. From a uniform circular disc a circular hole is cut touching the rim of the disc at the point A. Prove that the centre of gravity G of the remainder is on the rim of the hole when the diameter AB through the two centres is divided at G in "golden section," that is

$$AG^2 = AB \,.\, GB.$$

13. From a uniform circular disc is cut a circular hole touching the rim at one point. Show that as the radius of the hole is enlarged to approach that of the disc, the centre of gravity of the crescent tends to the middle point of a radius of the disc.

Hence show that the centre of gravity of a thin circular ring, of which the density varies as $\cos^2 \frac{1}{2}\theta$, where θ is the angle measured round the ring from a given point, bisects a radius of the ring.

14. A hollow cone, closed by a circular base of the same material, is placed within its circumscribing sphere. Show that its centre of gravity will coincide with the centre of the sphere, and it will rest in any position, if its semi-vertical angle α is given by

$$\sin \alpha = \tfrac{1}{6}(1 + \sqrt{7}).$$

15. The frustum of a cone with closed ends is formed of uniform sheet metal. The radius of the base is a, of the top b, the height is H and the slant side L. Prove that the height of the centre of gravity above the base is

$$\tfrac{1}{3}H \,.\, \frac{(a+2b)L + 3b^2}{(a+b)L + a^2 + b^2}.$$

16. A uniform sheet of metal forms the base of a pyramid, and n similar sheets are placed above it at equal differences of height extending to the vertex A, so that their edges lie on the contour of the pyramid. If D is the centre of gravity of the base, and G that of the $n+1$ sheets, prove that $AG/GD = 3 + 6/n$.

17. A cylindrical hole of radius a is bored out of a uniform solid hemisphere of radius R, the axis of the hole coinciding with the radius of the hemisphere at right angles to the plane base. Prove that the centre of gravity of the remaining solid is at a distance $\frac{3}{8}\sqrt{(R^2 - a^2)}$ from the base.

18. Three uniform beams of lengths $2a$, $2b$, $2c$ and weights $2A$, $2B$, $2C$ are supported, the middle one just overlapping the other two, by two props at the same level, so that all three are horizontal. Show how this may be done, and find the maximum distance between the props.

19. A heavy uniform wire is bent into a triangle ABC of area Δ, and is freely suspended from A. Prove that the inclination θ of BC to the horizon is given by

$$\tan \theta = \frac{(c-b)(a+b+c)^2}{4(2a+b+c)\Delta}.$$

20. A circular table of radius R is supported symmetrically by three vertical legs of equal weight. The radius of the circle circumscribing the feet is r. Prove that the ratio of the least weight on the edge of the table, which will overturn it on two legs, to the upward force on the opposite side which will have the same effect is $(2R+r)/(2R-r)$. Show also that the pressure on the two feet, about which the table begins to turn, is in the same ratio in the two cases.

21. A porter's truck is loaded fully and uniformly to the horizontal surface AC, A being over the handles and C over the wheels. When the handles are raised, show that the fraction of the load borne on the hands is $AB/2AC$, where B is the point in AC vertically over the wheels.

Show that the truck will give the greatest weight on the hands in a given position when it is so loaded that the point B is the middle point of AC.

22. A brick is placed with its flat side on the ground, and n bricks are placed vertically on top. The top brick is then pushed forward as far as possible in the direction of its length, then the first two together over the third, and so on in succession. Show that when all have been pushed forward in this way as far as possible, the top brick overhangs the one at the bottom by the distance

$$\frac{1}{2}L\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right),$$

where L is the length of each brick.

23. An arc of a circle of radius R subtends the angle 2α at the centre. A surface is formed by rotating this arc about its chord. Prove that the superficial area of the spindle formed is

$$4\pi R^2(\sin\alpha-\alpha\cos\alpha),$$

and that the volume of the spindle is

$$\tfrac{2}{3}\pi R^3[\sin\alpha(2+\cos^2\alpha)-3\alpha\cos\alpha].$$

24. A surface is formed by rotating the quadrant of a circle about the tangent at one of its extremities. Prove that the interior volume is

$$\tfrac{1}{6}R^3(4-\pi)(3\pi-4),$$

and that the area of the curved surface is $\pi(\pi-2)R^2$.

25. Prove that the distance OG between the centre of gravity of a uniform plane triangle and the centre of its circumscribing circle is

$$\tfrac{1}{3}R\sqrt{(1-8\cos A\cos B\cos C)},$$

where R is the radius of the circumscribing circle.

CHAPTER VII

COPLANAR FORCES IN GENERAL.

180. It has already been shown that, if R is the resultant of two parallel forces P, Q, the moment of R about any point O in the plane of the forces is equal to the sum of the moments of P and Q about the same point. The same theorem will now be proved for the case in which P, Q are intersecting forces.

Let P, Q, R be represented by sides and diagonal of the parallelogram $ABCD$. Two equal and opposite forces $\pm P$ acting along

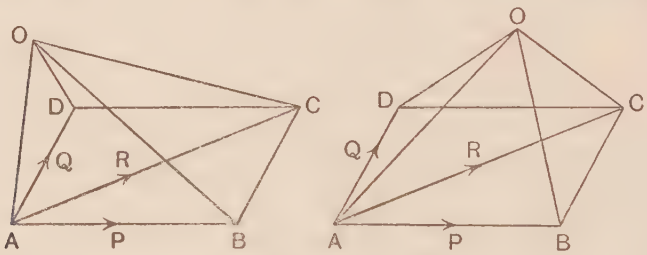

FIG. 100.

DC can be introduced, since they neutralize one another. In conjunction with P acting along AB they are equivalent to P acting along DC, together with the couple AB, CD. The moment of P along DC about O is represented by $2 \cdot ODC$, while the moment of the couple is represented by $ABCD = 2ADC$. Hence the moment of P in its original position is represented by $2 \cdot$ area $ODAC$. The moment of Q is $\pm 2 \cdot OAD$, according to the position of O. Hence the sum of the moments of P and Q is $2(ODAC \pm ODA) = 2 \cdot OAC$, which is the moment of R.

This principle can clearly be extended by the successive addition of any number of forces in the same plane, and thus, quite generally, the moment of the resultant of any system of coplanar forces is equal to the sum of the moments of the individual forces about the same point.

It will be noticed that, the proof being purely geometrical, the same property applies to any class of localized vectors in a plane, and not merely to forces.

181. The reduction of any plane system of forces acting on a rigid body, that is, at points rigidly connected, will now be explained.

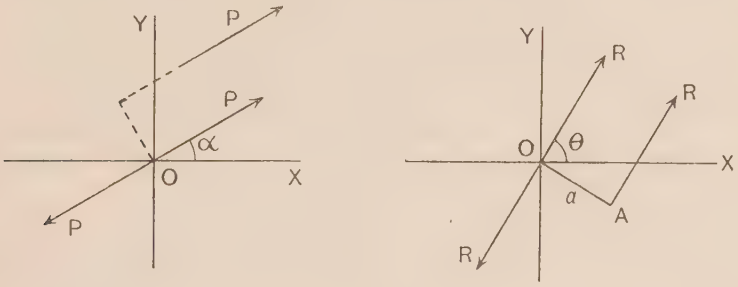

FIG. 101.

Through any fixed point of reference O let fixed perpendicular axes, OX OY be drawn. Let P denote any one of the system of forces, its line of action making the angle α with OX. By introducing two equal and opposite forces through O, equal in magnitude and parallel to P, the force P is transferred to O and a couple is added to the system. The effect of this couple is measured by its moment μ, and this moment is clearly the same as the moment about O of the force P in its original position, since the other member of the couple passes through O. When the whole system has been treated in this way, the result is to transfer all the forces parallel to themselves to the point O, this process being corrected by the addition of a resultant couple, of which the moment is represented by the sum of the moments about O of the forces in the given system, or $M=\Sigma\mu$.

The forces P transferred to O have a resultant R making an angle θ with OX, where

$$R\cos\theta=\Sigma P\cos\alpha, \quad R\sin\theta=\Sigma P\sin\alpha.$$

Let $M=Ra$ or $a=M/R$. Then the couple whose moment is M can be represented by two equal and opposite forces R acting on parallel lines at a distance a apart. This couple, completely defined by the magnitude and sign of M, can be moved in its plane until one component is directly opposed to the resultant R, and neutralizes it. The other component R of the couple is all that is left to represent the system of forces, which is thus reduced to a single force

acting along a definite line in the plane. Now this single resultant is the resultant R through O displaced parallel to itself through the distance a, to the right of the forward direction of R when M is positive, to the left when M is negative.

182. In general, neither $M=0$ nor $R=0$, and the system is reducible to a single force with a definite line of action not passing through O. There are, however, three special cases:

(a) If $M=0$ but R does not vanish, $a=0$, and the system reduces to a single force of which the line of action passes through O.

(b) If $R=0$ but M does not vanish, the system is equivalent to a couple, which cannot be reduced to a single force. This case involves a double condition, for if $R=0$,

$$\Sigma P\cos\alpha=\Sigma X=0,\quad \Sigma P\sin\alpha=\Sigma Y=0,$$

and the sum of the components in two distinct directions must vanish.

(c) If both $M=0$ and $R=0$, the couple and the single resultant both disappear, and the given system is in equilibrium. This case involves three conditions, namely,

$$\Sigma X=0,\quad \Sigma Y=0,\quad M=\Sigma\mu=0.$$

Ex. Forces 1, 2, 3, 4, 5, 6 act in order along the sides of a regular hexagon circumscribing a circle of radius 1. To reduce the system.

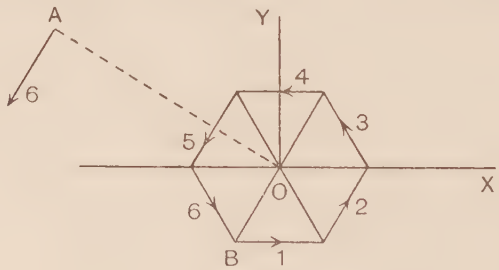

FIG. 102.

The axes being drawn as in the figure,

$$\begin{aligned} R\cos\theta&=1-4+(2-3-5+6)\cos 60^\circ\\ &=-3.\\ R\sin\theta&=(2+3-5-6)\sin 60^\circ\\ &=-6\sqrt{3}/2=-3\sqrt{3}.\\ R^2&=9+27=36,\quad R=6.\\ \cos\theta&=-\tfrac{1}{2},\quad \sin\theta=-\sqrt{3}/2,\quad \theta=240^\circ. \end{aligned}$$

The forces, transferred to O, have a resultant 6 along OB. Again, the moment of the system about O is

$$M = 1(1+2+3+4+5+6) = 21 = 3{\cdot}5\ R,$$

and is positive. Hence the system is reduced to a single force $R=6$ acting parallel to OB through the point A, where $OA = 3{\cdot}5$ is at right angles to OB.

183. It is now seen that any system of forces in a plane is equivalent either to a single force R or to a couple of moment M. In order to secure equilibrium from the system both these possibilities must be excluded. The necessary condition which excludes the couple is

$$M = \Sigma\mu = 0,$$

where $\Sigma\mu$ represents the sum of the moments of the forces of the system about a selected point O. But this essential condition does more than exclude the couple. As in case (a) above, it implies further that the single resultant force, to which the system may reduce, passes through the point O about which the moments μ are taken.

Hence the further conditions for equilibrium can be expressed in more than one way. As in case (c) above, they can be expressed in the form

$$\Sigma X = \Sigma Y = 0,$$

or in words, the sums of the components in any two distinct directions are zero. This is commonly the most convenient way of securing that $R=0$.

Alternatively, let moments be taken about a second point O' with the result expressed by

$$M' = \Sigma\mu' = 0.$$

When this second condition is satisfied, the resultant R must pass not only through O but also through O', and if it exists its line of action is OO'. Hence the conditions for equilibrium, requiring $R=0$, can now be completed in one of two ways. The sum of the components at right angles to OO' being necessarily zero, it will be sufficient if the sum of the components in any other direction also vanishes. Or again, since the system is now equivalent to a force R acting along OO', the moment of the system about any point on OO' is necessarily zero, while the moment about a third point O'' not situated on OO' can only vanish when $R=0$. The third required condition can therefore be expressed in this way.

184. The conditions for the equilibrium of a coplanar system of forces can now be summarized thus :

(*a*) The moment about one point vanishes, and the sums of the components in two directions also vanish.

(*b*) The moments about two points vanish, and the sum of the components in any direction not at right angles to the line joining these points also vanishes.

(*c*) The moments about three points not situated on one straight line vanish.

Any one of these combinations of three conditions will secure the equilibrium of the system. In the case of very simple systems, of two or three forces, the general conditions can be replaced by simple rules. Thus two forces in equilibrium must obviously be equal and opposite, and act on the same straight line.

185. When three forces are in equilibrium, any one of the three must be the reversed resultant of the other two. Hence the forces must lie in one plane. If two of the forces are parallel, the third must also be parallel to them. If two of the forces intersect at a finite distance the third must also pass through the point of intersection. In this case the further necessary conditions can be expressed in a simple and convenient form.

If P, Q, R are the three forces in equilibrium, R is equal and

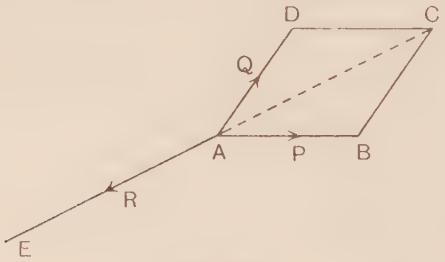

FIG. 103.

opposite to AC, the diagonal of the parallelogram $ABCD$. Hence

$$\frac{P}{AB}=\frac{Q}{AD}=\frac{Q}{BC}=\frac{R}{AC},$$

or
$$\frac{P}{\sin ACB}=\frac{Q}{\sin BAC}=\frac{R}{\sin ABC},$$

or
$$\frac{P}{\sin EAD}=\frac{Q}{\sin BAE}=\frac{R}{\sin DAB},$$

since $\sin ACB = \sin CAD = \sin EAD$. In words, each of the three forces is proportional to the sine of the angle between the other two. This form of statement is often more useful than the equivalent statement of the triangle of forces, according to which the forces are proportional to the parallel sides of a triangle ABC, taken in order.

186. *Ex.* A thin non-uniform bar of length $a+b$, with centre of gravity at distances a, b from the two ends, is suspended from a fixed point by light strings of length l_1, l_2 attached to its ends. To find the position of equilibrium and the tensions of the strings.

The rod AB is in equilibrium under three forces, the tensions T_1, T_2 and its weight W acting vertically through the centre of gravity G. As

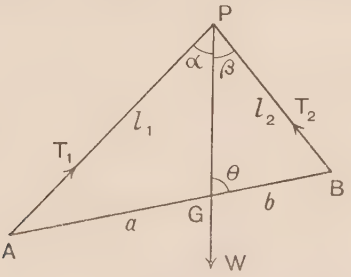

FIG. 104.

T_1, T_2 meet in the point of suspension P, PG must be vertical, and further,

$$\frac{T_1}{\sin\beta} = \frac{T_2}{\sin\alpha} = \frac{W}{\sin(\alpha+\beta)},$$

where α, β are the inclinations of the strings to the vertical. If $\theta = PGB$ is the inclination of the rod to the vertical, the whole problem is clearly solved when θ, α and β have been found, and this is purely a question of geometry.

Thus, if $PG = x$,

$$l_1^2 = a^2 + x^2 + 2ax\cos\theta,$$
$$l_2^2 = b^2 + x^2 - 2bx\cos\theta,$$

whence

$$x^2 = b\,.\,\frac{l_1^2 - a^2}{a+b} + a\,.\,\frac{l_2^2 - b^2}{a+b},$$

$$2x\cos\theta = \frac{l_1^2 - a^2}{a+b} - \frac{l_2^2 - b^2}{a+b},$$

so that x and θ are determined. Since

$$\sin\alpha = \frac{a}{l_1}\sin\theta, \quad \sin\beta = \frac{b}{l_2}\sin\theta,$$

the angles α, β are also determined, and the problem is completely solved.

187. *Ex.* The conditions of the previous problem are varied by uniting the ends of the strings at P and passing the loop over a smooth

peg. Since the peg is smooth and exercises no action along the string, a uniform tension T is communicated along the string and $\beta = \alpha$.

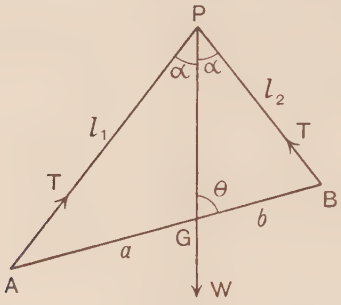

FIG. 105.

The statical condition becomes

$$\frac{T}{\sin\alpha} = \frac{W}{\sin 2\alpha},$$

or $T = \frac{1}{2}W \sec\alpha$. The combined length $l_1 + l_2$ being given,

$$\frac{\sin\theta}{\sin\alpha} = \frac{l_1}{a} = \frac{l_2}{b} = \frac{l_1 + l_2}{a+b} = k,$$

so that $l_1 = ak$, $l_2 = bk$, $\sin\theta = k\sin\alpha$, where k is known.

It only remains to determine α, and for this purpose

$$(a+b)^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos 2\alpha$$
$$= k^2\{(a+b)^2 - 4ab\cos^2\alpha\},$$

so that $\cos^2\alpha = (k^2-1)(a+b)^2/4abk^2$,

and $\cos^2\theta = 1 - k^2(1-\cos^2\alpha)$
$= (k^2-1)(a-b)^2/4ab.$

188. *Ex.* A hollow cylinder of radius b and weight W, with open ends, stands on a level surface. Two smooth spheres, each of radius $a\,(a < b < 2a)$ and weight w, are placed inside. To find the reactions.

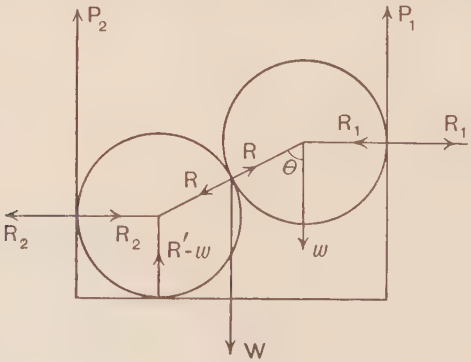

FIG. 106.

The vertical plane which contains the axis of the cylinder and the centres of the spheres is a plane of symmetry ; let θ be the angle between the line of centres and the vertical, so that

$$2a \sin\theta = 2b - 2a.$$

The upper sphere is in equilibrium under its weight w, the reaction R_1 with the cylinder and the reaction R with the lower sphere. The lower sphere is in equilibrium under the reaction R with the upper sphere, the reaction R_2 with the cylinder, the reaction R' with the ground and its weight w. In both cases the forces act through the centres of the spheres, and in the latter the forces R' and w, acting on the same vertical radius, combine in the single force $R' - w$. The statical conditions give immediately

$$\frac{R}{1} = \frac{R_1}{\sin\theta} = \frac{w}{\cos\theta},$$

$$\frac{R}{1} = \frac{R_2}{\sin\theta} = \frac{R' - w}{\cos\theta},$$

or $$R = w\sec\theta, \quad R_1 = R_2 = w\tan\theta, \quad R' = 2w.$$

The pressure of the ground on the cylinder is distributed round the lower rim, but may be replaced, owing to the symmetry, by two vertical forces P_1, P_2. The spheres being smooth, the forces acting on the cylinder are then P_1, P_2 (vertical), R_1, R_2 (horizontal) and the weight W on the axis. The conditions of equilibrium, obtained by considering (a) the vertical components and (b) the moments about the point of contact of the spheres, are

$$P_1 + P_2 = W,$$

$$(P_1 - P_2)b - (R_1 + R_2)a\cos\theta = 0,$$

or $$P_1 - P_2 = 2wa\sin\theta/b = 2w(b-a)/b,$$

whence $$P_1 = \tfrac{1}{2}W + w\left(1 - \frac{a}{b}\right), \quad P_2 = \tfrac{1}{2}W - w\left(1 - \frac{a}{b}\right).$$

Now the pressure of the ground on the cylinder, or of the cylinder on the ground, cannot be negative without some form of attachment. Hence, if the cylinder is to stand freely on the ground without overturning,

$$W > 2w\left(1 - \frac{a}{b}\right),$$

in order that P_2 may be positive.

189. In general the problems of statics require the reduction of a system of forces to a simpler equivalent, but this is not always the case, especially when internal stresses are in question. Something like the reverse process is illustrated in the following example.

Ex. A uniform rod is held horizontally at one end. To find the action of forces at any point of its length.

Let the rod AB, of length a and linear density ρ, be fixed at A. Let P be the point at which the action is required, where $AP = x$. The

part PB, of length $a-x$ and weight $w=g\rho(a-x)$, is in equilibrium under gravity and the action of forces at P communicated by the part

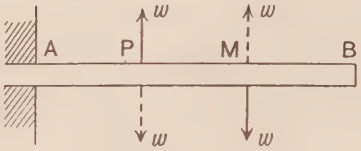

FIG. 107.

AP of the rod. The action of gravity on PB is represented by the weight w at M, the middle point of PB, so that $PM=\frac{1}{2}(a-x)$. This force could be balanced by an upward vertical force w at M. The action at P must be equivalent to this upward force at M. It cannot be represented by any single force at P.

Add equal and opposite vertical forces w at P. The required action at P can now be represented by the upward force w at P and a couple. The moment of this couple is $w.PM=\frac{1}{2}g\rho(a-x)^2$, and this is the *bending moment* at P. As it must be communicated by forces at P, the rod must have a cross-section of finite area on which they can act.

When it is understood that a force and a couple are necessary to preserve equilibrium, the force can be determined by considering the components, and the bending moment by considering the moments about the point, of the forces acting on one side of the point. The force itself and the corresponding opposite reaction constitute the *shearing stress* at the point.

190. The reduction of any system of coplanar forces can be effected by graphical methods, and the conditions of equilibrium can also be considered from the graphical point of view.

Two diagrams are necessary. In the first the forces of the system, P, Q, R, ... are represented in magnitude, direction and position. In the second the same forces are represented in magnitude and direction by the consecutive sides 12, 23, 34 ... of a polygon, called the force polygon. In the plane of this polygon an arbitrary point O is taken and joined to the vertices 1 2 3 4 ....

On the line of action of the force P in the first diagram an arbitrary point A is taken, and through it are drawn lines AX, AB respectively parallel to 01, 20. On these lines of action are placed neutralizing pairs of forces 10, 01, 20, 02, by which the whole system is unaffected. But the forces 01, P, 20, which act at A, being equal and parallel to the sides of the triangle 012 in order in the force diagram, are in equilibrium. Hence the effect of this operation is to replace the force P by 10 acting along XA and 02

acting along BA. These substituted forces naturally depend on the choice of the point O in the plane of the force diagram.

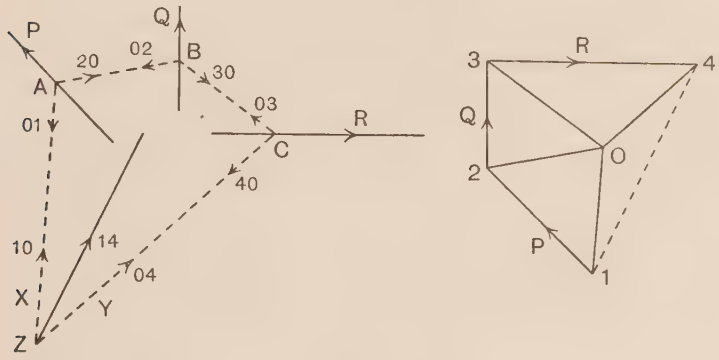

Fig. 108.

Let AB meet the line of action of Q in the point B, and draw BC parallel to 30. The equal and opposite forces 30, 03 being placed on BC, the three forces 02, Q, 30, which act at B and are equal and parallel to the sides of the triangle 023 in order, are in equilibrium. Hence at this stage the forces P and Q are replaced by 10 acting along XA and 03 acting along CB.

191. This process can clearly be extended to any number of forces. If R is the last force included, BC meets the line of action of R in C and CY is drawn parallel to 40. On this line the neutralizing forces 40, 04 are placed. The forces 03, R and 40, acting through C, are in equilibrium, because they are equal and parallel to the sides of the triangle 034 in order. Hence the system of forces P, Q, R ..., is finally replaced by a pair of forces 10 acting along XA and 04 acting along YC. The rectilinear figure $XABCY$ is called a *funicular polygon* of the system of forces. Its construction can be varied indefinitely, owing to the arbitrary choice of the point O and the point A.

In general, the lines XA, YC, which are parallel to 10, 04, are not parallel to one another, but meet in a point Z. The resultant of the system, or of the forces 10, 04, therefore passes through the point Z. Moreover, the resultant of the forces 10, 04 is represented in magnitude and direction by 14 in the force diagram. The resultant of the system is therefore obtained by drawing a force equal and parallel to 14 through Z.

By exception, the force polygon may be closed, in which case

the first and last points, here 1 and 4, coincide. Then the forces 10, 04 are equal and opposite. The lines XA, YC are therefore parallel or coincident. If they are parallel and distinct, the original system of forces reduces to a couple. If they are coincident, so that the funicular polygon is closed, the forces 10, 04 neutralize one another and the system is in equilibrium.

Hence the equilibrium of the system requires that both the force polygon and the funicular polygon shall be closed. If the force polygon alone is closed, the first and last sides of the funicular polygon are parallel and the system is equivalent to a couple. If the force polygon is not closed, it can be closed by the addition of a side, which represents the single resultant force in magnitude and direction, while the line of action of this force can be determined by constructing a funicular polygon, whose first and last sides intersect on this line of action.

Ex. Deduce a simple construction for the resultant of two parallel forces.

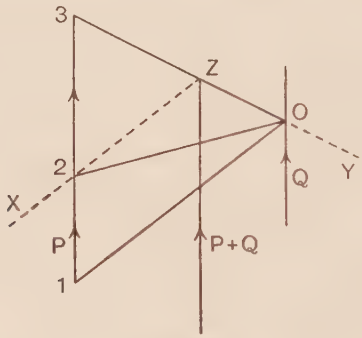

FIG. 109.

Let $P=12$ and Q be the parallel forces. The force diagram in this case is the straight line 123, where $23=Q$. Take the pole O on the line of action of Q. A funicular polygon is constructed in the same diagram by the sides $2X$ parallel to $O1$, $O2$ in position, and OY, which is the line $3O$ produced. Hence the resultant is the force $P+Q$, equal and parallel to 13, drawn through the point Z, in which the parallel to the line $1O$ through the point 2 cuts the line $3O$.

192. *Ex.* A light string, with weights W_1, W_2, W_3 attached at points along it, is suspended from two fixed points. To consider the conditions of equilibrium.

Let the end of the string E be fixed at a horizontal distance $AN=a$ from the end A and at depth $NE=b$ below it. Let the string be divided at the points of attachment of the weights into lengths $AB=l_1$, $BC=l_2$,

$CD = l_3$, $DE = l_4$. These sections, in which the corresponding tensions are T_1, T_2, T_3, T_4, are inclined to the vertical at the angles α_1, α_2, α_3, α_4 respectively.

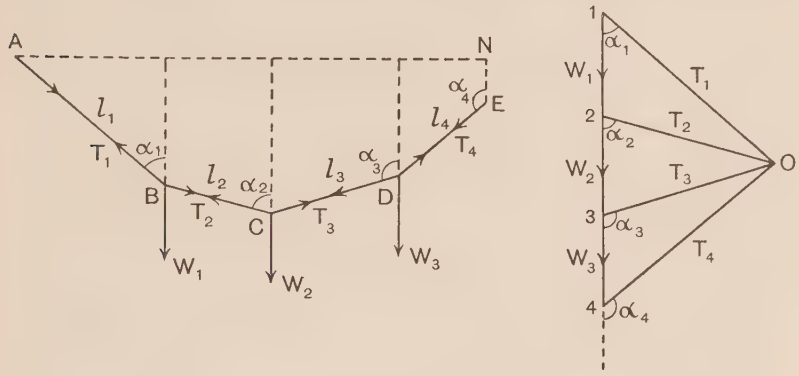

FIG. 110.

The horizontal and vertical projections of these lines give immediately the two conditions,

$$l_1 \sin \alpha_1 + l_2 \sin \alpha_2 + l_3 \sin \alpha_3 + l_4 \sin \alpha_4 = a,$$
$$l_1 \cos \alpha_1 + l_2 \cos \alpha_2 + l_3 \cos \alpha_3 + l_4 \cos \alpha_4 = b.$$

A force diagram is now constructed. The weights $W_1 = 12$, $W_2 = 23$, $W_3 = 34$ form one vertical line 14. The lines $1O$, $2O$ meeting in O are drawn parallel to AB, BC. Since they are parallel to T_1, T_2, which are in equilibrium with W_1 at B, it follows that $T_1 = O1$ and $T_2 = 2O$. At the point C, T_3 is in equilibrium with W_2 and $T_2 = O2$. Hence T_3 is represented, in magnitude and direction, by $3O$. Similarly at D, T_4 is in equilibrium with W_3 and $T_3 = O3$, so that T_4 is represented, in magnitude and direction by $4O$. The opposite member of this stress, $O4$, is borne by the fixed point E.

The force diagram is now complete, and it will be observed that the form of the string is a funicular polygon in the above sense, thus explaining the use of the term in its more general application.

The remaining conditions can be deduced at once from the force diagram. For clearly

$$T_1 \sin \alpha_1 = T_2 \sin \alpha_2 = T_3 \sin \alpha_3 = T_4 \sin \alpha_4,$$
$$W_1 = T_1 \cos \alpha_1 - T_2 \cos \alpha_2, \quad W_2 = T_2 \cos \alpha_2 - T_3 \cos \alpha_3,$$
$$W_3 = T_3 \cos \alpha_3 - T_4 \cos \alpha_4,$$

whence
$$\frac{\cot \alpha_1 - \cot \alpha_2}{W_1} \quad \frac{\cot \alpha_2 - \cot \alpha_3}{W_2} = \frac{\cot \alpha_3 - \cot \alpha_4}{W_3}.$$

These two relations, together with the two conditions first written down, must be satisfied by the four angles, which determine the position of equilibrium of the string. When these are known the tensions are easily deduced.

193. A framework can be constructed of straight rigid rods freely jointed together at their extremities. The connexions may be considered to take the form of small smooth pins passing through the rods and incapable of offering any resistance to the rods turning about them. When such a framework is suitably constructed it is *simply rigid*. From this condition it may depart by being freely deformable or by being over-rigid. In the former case it needs the addition of further connecting rods to make it rigid, in the latter it will remain rigid even when one or more rods are removed. When a system of external forces is applied at the connecting pins, a set of stresses is set up in the rods of the framework. The stresses, either tensions or compressions (thrusts), are in general determinate when the frame is simply rigid; when the frame is over-rigid equilibrium is, of course, still possible, but the distribution of the stresses can no longer be definitely assigned.

Three rods in the form of a triangle give the simplest case of a rigid frame, and the triangle is the simplest element in a more complicated arrangement. When four rods form a quadrilateral the frame is not rigid at all. It can be made simply rigid by adding one diagonal rod. The length of the other diagonal is now fixed, and if this diagonal is represented by a sixth rod, which is not required for rigidity, the frame becomes over-rigid.

Ex. A quadrilateral frame $ABCD$, with the diagonal rod BD, is in equilibrium under opposing forces P applied at A and C. To represent the stresses graphically.

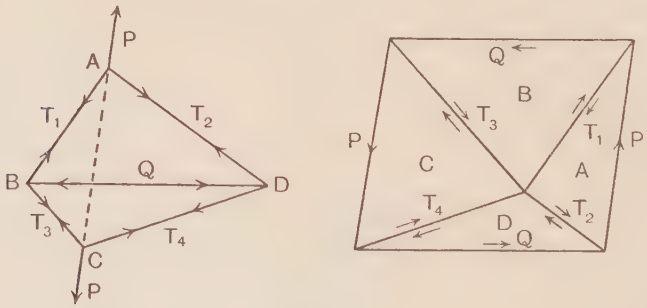

FIG. 111.

It is obvious that the applied forces P act along the line AC. Equilibrium at A and C requires tensions T_1, T_2, T_3, T_4 in the outer rods, and equilibrium at B and D then requires a thrust Q in BD.

Construct the triangle A with one side representing the force P in magnitude and direction and the other sides parallel to T_1, T_2, which

act at the point A. These sides represent T_1, T_2 in magnitude also. On the side T_1 of the triangle A construct the triangle B with sides parallel to the forces Q, T_3, which act at the point B. These sides represent Q, T_3 in magnitude also. On the side T_3 of triangle B construct the triangle C with sides parallel to the forces P, T_4 which act at the point C. These sides represent P, T_4 in magnitude also. The triangle D can now be completed by joining the extremities of T_4, T_2 and the base of this triangle is clearly equal and parallel to Q, because the equal and parallel sides P make the whole force diagram a parallelogram. The sides of the triangle D thus represent the forces acting at the point D, which are accordingly in equilibrium, both in magnitude and direction. The stresses are therefore represented in the force diagram. It will be noticed that the consistency of the representation is a natural consequence of the conditions of equilibrium being observed in the external forces.

194. The use of the force diagram is illustrated in the following slightly more complicated example.

Ex. The angles in the frame $ABCDE$, in which BE, BD are joined by rods, are either 90° or 45°. The force P is applied at A parallel to BE or CD, and equilibrium is maintained by a force Q at D in the direction DE and a force R at C. To determine Q (in magnitude), R (in magnitude and direction) and the stresses in the rods.

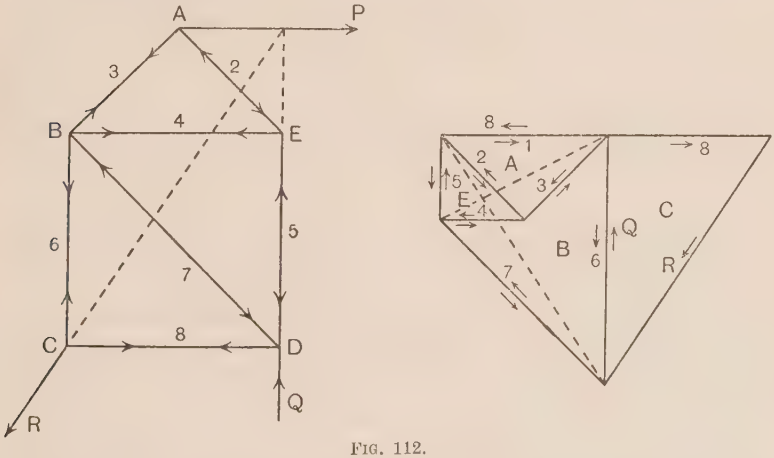

FIG. 112.

The rods in the framework are numbered 2 to 8, and the stresses are given the corresponding numbers in the force diagram. The force P being represented by the side 1, the three forces which give equilibrium at A are given in the force diagram by the parallel sides of the triangle 123. Similarly, the three forces which give equilibrium at E are given by the parallel sides of the triangle 245. At B four forces intersect, but two of them are represented jointly by joining the ends of 3 and 4.

From these ends the lines 6 and 7 are drawn in the appropriate directions, so that these forces are determined in magnitude. Similarly at D four forces intersect, but the resultant of 5 and 7 is given by joining their extremities, and as the directions of 8 and Q are given, it becomes evident that 8 is equal to 1 or P, and that Q is equal to 6. Finally, the three forces in equilibrium at C are R, 6 or Q and 8 or 1, which are represented by the sides of the triangle C in the force diagram.

The stresses are marked in the frame in the sense of the action of each rod on the adjacent pin. In the force diagram the forces corresponding to the actions on each pin form the continuous circuit in one direction of a triangle or polygon, this being the condition of equilibrium of the intersecting forces. The geometry of the force diagram being in this instance particularly simple, the magnitude of the stresses can be immediately read off. Thus in the rod 2 there is a thrust, and in rod 3 a tension, of magnitude $P/\sqrt{2}$. In rod 4 there is a tension, and in rod 5 a thrust, of magnitude $\frac{1}{2}P$. In rod 6 there is a tension $\frac{3}{2}P$, in rod 7 a thrust $P\sqrt{2}$, and in rod 8 a tension P. The force Q is $\frac{3}{2}P$, being equal to the tension in rod 6, and the force R is $\frac{1}{2}P\sqrt{13}$, being defined by its components P and $\frac{3}{2}P$.

The process assures the equilibrium of each joint separately, and therefore of the frame as a whole. Since the stresses neutralize one another in pairs of forces, the external forces alone must also be in equilibrium. This is easily verified in the present case, because the forces P, Q, R are represented by the sides of the triangle C, and the force R, having the components P and $\frac{3}{2}P$, clearly passes through the intersection of P and Q. The nature of the forces Q and R has here been deduced by the purely graphical method. In other similar cases it may be more convenient, as it would have been quite possible here, to ascertain the necessary relations between the external forces before investigating the internal stresses.

195. When a body is placed in a certain situation it is a matter of common observation that the body will remain at rest even when the applied forces obviously acting, together with the normal reactions of surfaces in contact with it, are not such as to produce equilibrium. As the normal reactions of fixed surfaces can be evoked without limit in such measure as equilibrium may require, the latent forces which must be acting in order to preserve equilibrium may be considered to be tangential to the surfaces in contact. This is the fundamental property of frictional forces, which act always at right angles to a normal pressure between surfaces.

Thus, when a body is placed on a fixed inclined plane, the normal pressure R prevents motion in any direction other than down the plane. In the absence of any other force the weight will produce an acceleration $W \sin\theta/m = g \sin\theta$ down the plane, where θ is the

inclination of the plane. If, on the contrary, the body remains at rest, equilibrium must be ascribed to the action of a force acting up

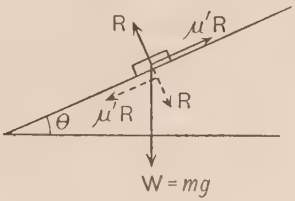

FIG. 113.

the plane. This force being denoted by $\mu' R$, the conditions for equilibrium are

$$W = \frac{R}{\cos\theta} = \frac{\mu' R}{\sin\theta},$$

whence $$\mu' = \tan\theta.$$

If now the body be placed on a horizontal plane, which is then slowly tilted, unless equilibrium is first destroyed by some other cause (*e.g.* by the body overturning), a point will be reached when the body will begin to slip down the plane. Let this be on the point of happening when $\theta = \alpha$, where $\tan\alpha = \mu$. The possibility of equilibrium is therefore limited in this case by the conditions

$$\theta < \alpha, \quad \mu' < \mu.$$

Both the normal pressure R and the frictional force $\mu' R$ are subject to the law of action and reaction. They are components of stresses, of which the opposite components act on the other body in contact with the first. While, however, the pressure R can be evoked to any extent necessary to prevent relative motion, the friction $\mu' R$ cannot exceed a definite ratio to R. When equilibrium would require this ratio to be exceeded, motion ensues. The limiting ratio μ is called the *coefficient of friction* of the surfaces in contact, and the corresponding angle α, where $\tan\alpha = \mu$, is called the *angle of friction.*

196. The following laws may be stated for solid friction :

(1) Friction acts at every point of contact between two solid bodies when there is normal pressure between the surfaces.

(2) Friction acts in the tangent plane at the point of contact and in that direction in this plane which is opposite to the direction of slipping, if it actually occurs, or to the direction in which one surface would slip over the other if friction were absent.

(3) When slipping occurs, the force of friction is μR, where R is the total pressure at the point (or rather area) of contact. The coefficient of friction μ depends only on the nature of the surfaces in contact. It is independent of the relative velocity of slipping, and of the area over which the total pressure R may be distributed.

(4) If a force $\mu' R$, where $\mu' \leq \mu$, will suffice to prevent slipping, then slipping will not take place, and equilibrium will result, provided that the other circumstances of the problem are compatible with equilibrium.

These laws are open to immediate criticism when applied to the actual substances of nature. Yet they may be taken quite usefully to define an ideal solid friction by properties which real materials possess in a greater or less degree. This is consistent with the use of a rigid body, implying a perfectly hard surface, which does not exist, for the purposes of abstract reasoning. When slipping occurs between real bodies, the coefficient μ is not independent of the velocity and has a notably smaller value than in static friction before equilibrium is overcome. The actual value of μ in any case is very uncertain. Thus between two polished glass surfaces the value depends entirely on the state of chemical purity of the surfaces, and hence on the manner of cleaning and, when exposed to the air, on the time elapsing after this process.

197. *Ex.* A weight W is placed on an inclined plane, the inclination θ being greater than the angle of friction α. Find the least and greatest

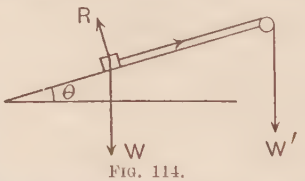

FIG. 114.

weights W' which will maintain equilibrium when attached over a smooth pulley.

If W_1 is the least weight preventing W from slipping down, $W_1 + \mu R$ is the force acting on W up the plane. If W_2 is the greatest weight when W is on the point of moving up the plane, the friction μR acts down the plane, so that $W_2 - \mu R$ is the force acting on W up the plane. Hence the conditions of limiting equilibrium are

$$W = \frac{W_1 + \mu R}{\sin\theta} = \frac{R}{\cos\theta},$$

$$W = \frac{W_2 - \mu R}{\sin\theta} = \frac{R}{\cos\theta},$$

or
$$W_1 = W \sin\theta - \mu R = W(\sin\theta - \mu\cos\theta),$$
$$W_2 = W \sin\theta + \mu R = W(\sin\theta + \mu\cos\theta),$$
where $\mu = \tan\alpha$. Equilibrium is possible when $W_1 < W' < W_2$, or
$$W \sin(\theta - \alpha) \leq W' \cos\alpha \leq W \sin(\theta + \alpha).$$

198. *Ex.* A uniform ladder of weight W is placed on horizontal ground against a vertical wall, the coefficients of friction at the ground and wall being μ_1, μ_2. A weight w is placed on the ladder (of length a) at a distance b from the bottom. Find the position of limiting equilibrium.

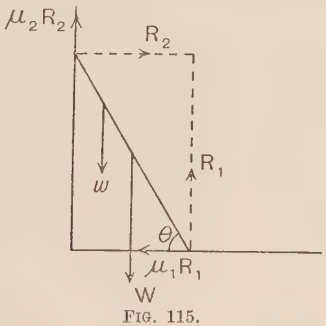

FIG. 115.

As the ladder in limiting equilibrium is on the point of slipping down, the frictional forces $\mu_1 R_1$ and $\mu_2 R_2$ act in the directions shown. The horizontal and vertical components of the forces give the conditions
$$R_2 = \mu_1 R_1,$$
$$R_1 + \mu_2 R_2 = W + w.$$

If θ is the inclination of the ladder, by taking moments about its lower end,
$$aR_2(\mu_2\cos\theta + \sin\theta) = (\tfrac{1}{2}aW + bw)\cos\theta.$$

But
$$R_1 = \frac{R_2}{\mu_1} = \frac{R_1 + \mu_2 R_2}{1 + \mu_1\mu_2} = \frac{W + w}{1 + \mu_1\mu_2}.$$

Hence
$$\frac{a\mu_1(W + w)}{1 + \mu_1\mu_2}(\mu_2 + \tan\theta) = \tfrac{1}{2}aW + bw,$$

or
$$\tan\theta = \frac{\tfrac{1}{2}aW + bw}{a\mu_1(W + w)} \cdot (1 + \mu_1\mu_2) - \mu_2$$
$$= \frac{1 - \mu_1\mu_2}{2\mu_1} + \frac{b - \tfrac{1}{2}a}{a} \cdot \frac{w}{W + w} \cdot \frac{1 + \mu_1\mu_2}{\mu_1}.$$

If the ladder, with the load w, is in a position of limiting equilibrium, θ is a minimum. If b is diminished by lowering w, this minimum is reduced and the ladder will remain in equilibrium. If w is caused to ascend, the ladder will slip down, unless placed closer to the wall.

199. *Ex.* A uniform solid hemisphere, of radius a, rests on two parallel rods, at the distance a apart in the same horizontal plane.

To find the limiting position of equilibrium, the coefficient of friction between the rods and the hemisphere (curved surface) being $\mu = \tan\alpha$.

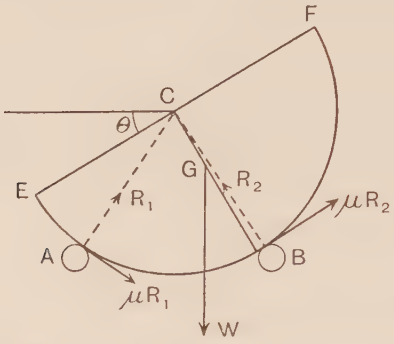

FIG. 116.

The points of contact, A, B form an equilateral triangle with C, the centre of the hemisphere. In the limiting position let the diameter EF make the angle θ with the horizon. The reactions R_1, R_2 act along the radii AC, BC and the limiting frictional forces μR_1, μR_2 act tangentially in the sense indicated, so as to prevent rotation about C. The weight W acts through G, where $CG = \frac{3}{8}a$ is inclined to the horizon at the angle $90° - \theta$.

The horizontal and vertical components of the forces give the conditions:

$$(R_1 - R_2)\cos 60° + \mu(R_1 + R_2)\cos 30° = 0,$$

$$(R_1 + R_2)\cos 30° + \mu(R_2 - R_1)\cos 60° = W,$$

whence
$$\frac{R_2 - R_1}{\mu\sqrt{3}} = \frac{R_2 + R_1}{1} = \frac{2W}{(\mu^2+1)\sqrt{3}},$$

or
$$\frac{R_1}{1 - \mu\sqrt{3}} = \frac{R_2}{1 + \mu\sqrt{3}} = \frac{W}{(\mu^2+1)\sqrt{3}}.$$

The third condition of equilibrium, by taking moments about C, is

$$\mu(R_1 + R_2)a = \tfrac{3}{8}W \,.\, a\sin\theta,$$

so that
$$\sin\theta = \frac{8\mu}{3} \cdot \frac{R_1 + R_2}{W} = \frac{16\mu}{3\sqrt{3}(\mu^2+1)} = \frac{8}{3\sqrt{3}}\sin 2\alpha.$$

When the point E is displaced to A, the conditions of the problem are altered. The limiting position of present interest is thus limited so that

$$\theta < 60°, \quad \sin\theta < \frac{\sqrt{3}}{2}, \quad \sin 2\alpha < \frac{9}{16},$$

whence $\alpha < 17°{\cdot}1$, $\mu = \tan\alpha < 0{\cdot}308$. If the coefficient of friction μ has a smaller value than this, EF can be rotated from the horizontal to a definite position, short of CA, which has been determined and beyond which equilibrium fails. If μ has a greater value, the hemisphere

will rest in any position in which it is placed with its curved surface on both the rods and its plane face parallel to the rods.

200. In the absence of friction the positions of equilibrium are generally, though not always, definite and distinct. With friction equilibrium is possible in a continuous series of positions. Such a series is terminated in either direction by one of several causes :

(*a*) The geometrical conditions, the effect of which is usually obvious. Thus, when a ladder is placed against a wall, the series clearly comes to an end when the bottom of the ladder comes into contact with the wall.

(*b*) The statical conditions apart from friction. If for example the reaction at a stop, which can only apply a pressure, is required to change sign, in order to preserve equilibrium, then equilibrium will cease to be possible.

(*c*) When the series of positions of equilibrium is not interrupted by one or other of these causes, it continues up to the limit at which friction becomes insufficient to maintain equilibrium.

Thus, when a uniform right circular cylinder is placed with its base on a plane, which is slowly tilted from the horizontal position, friction will prevent it from sliding down the plane until the inclination of the plane is $\theta = \alpha$, where $\mu = \tan\alpha$ is the coefficient of friction between the plane and the base of the cylinder. Before this position has been reached, however, equilibrium may be broken by the cylinder overturning. This will happen when the centre of gravity

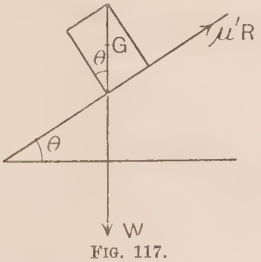

FIG. 117.

G is vertically over the lowest point of the base. The total reaction R is the resultant of the positive normal pressure distributed over the base, and cannot act at a point outside the base. In the limiting position before overturning, the forces W, R and $\mu' R$ therefore intersect at the lowest point of the base. If d is the diameter and h the height of the cylinder, $\tan\theta = d/h$ in this position. If $d < \mu h$, this position will be reached first and the cylinder will fall over.

If $d > \mu h$, this will not occur, but when $\theta = \alpha$, the cylinder will slide down the plane.

201. *Ex.* The conditions in the example of § 199 are changed by weighting the hemisphere in such a way that $CG = ka$, the centre of gravity G being still on the axis of the hemisphere.

In the position of limiting friction the forces are as before, except that the weight W is displaced in position. Hence

$$\frac{R_1}{1-\mu\sqrt{3}} = \frac{R_2}{1+\mu\sqrt{3}} = \frac{W}{(\mu^2+1)\sqrt{3}},$$

as before, but moments about C give the condition

$$\mu(R_1+R_2)a = W \,.\, ka \sin\theta,$$

so that θ is given by

$$\sin\theta = \frac{2\mu}{k(\mu^2+1)\sqrt{3}} = \sin 2\alpha / k\sqrt{3}.$$

This limit will occur before that set by the geometrical condition $\theta = 60°$ if $\sin 2\alpha < \frac{3}{2}k$. But it will not be actually reached unless the reaction R_1 remains positive, or $\mu = \tan\alpha < 1/\sqrt{3}$, $\alpha < 30°$, $\sin 2\alpha < \frac{1}{2}\sqrt{3}$. This condition is covered by the former if $k < 1/\sqrt{3} < 0{\cdot}577$. If, then,

$$\sin 2\alpha < \tfrac{3}{2}k < \tfrac{1}{2}\sqrt{3},$$

as in the case discussed, where $k = \frac{3}{8}$, equilibrium will first fail in the position of limiting friction. If $\alpha < 30°$, but $k > 0{\cdot}577$, the series of equilibrium positions will also end through the failure of friction.

If $\alpha = 30°$, the limiting position is given by

$$ka \sin\theta = a \sin 2\alpha/\sqrt{3} = \tfrac{1}{2}a.$$

In this position $R_1 = 0$ and the centre of gravity G is vertically over the second rod, a position only possible, with $\theta < 60°$, when $k > 1/\sqrt{3}$.

If the hemisphere be now turned still further, G will fall beyond the supporting rod and equilibrium will fail by the hemisphere overturning. No increase in the coefficient of friction will prevent this from happening, and so the conditions

$$k > 1/\sqrt{3}, \quad \mu > 1/\sqrt{3}, \quad \sin\theta = 1/2k,$$

define a position of limiting equilibrium in which the series ends from causes independent of friction.

202. In developing the consequences of the laws of dynamics it has been seen that the motions occurring in a system of particles can be investigated by treating each particle individually, but that there exist certain principles, namely those of momentum and energy, whereby certain results applying to the system as a whole can be derived immediately. It is natural to inquire whether anything corresponding with these principles is available in statics.

In the notation already used (§ 115),

$$\Sigma mf = \Sigma X + \Sigma R_x.$$

Here, as before, the sum of the internal stresses $\Sigma R_x = 0$. But further, in statical equilibrium, every acceleration $f = 0$. Hence

$$\Sigma X = 0.$$

This expresses no more than that the sum of the external, applied forces resolved in any chosen direction vanishes, and adds nothing to the conditions of equilibrium already established. But it is well to bear in mind that in this condition the internal forces R_x do not appear, not because they do not exist, but because, occurring in equal and opposite pairs, they neutralize one another. For this reason, a special case of Newton's third law, they can be largely ignored in the rigid, incompressible bodies of statics, but they are of paramount importance in the consideration of deformable, elastic bodies.

It may appear, on the other hand, that where a stationary system in equilibrium is concerned, there can be nothing corresponding with the principle of energy, since in this principle actual displacements are involved. This, however, is far from being true.

203. Any material system can be ultimately resolved into a number (generally very large) of small particles m. Each such particle is subject to forces which can be reduced to an applied force P and an internal action R. These two forces, in the state of equilibrium, must be equal and opposite. If the particle be displaced

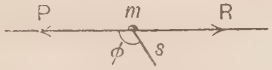

FIG. 118.

a finite distance s, these forces will be changed in general and equilibrium will be destroyed. But if s be indefinitely small, the change in P and R will in general be of the same order as s, and it is possible to write

$$P \cos \phi \,.\, s + R \cos (180° - \phi) \,.\, s = 0,$$

where ϕ is the angle between P and the displacement s. This is a mere identity if P and R are given their values in the undisturbed position. When the terms of the equation are interpreted as the work done in the displacement, changes bearing a finite ratio to s must be admitted in P and R, and the sum of the terms will bear a finite ratio to s^2. Had the particle been out of equilibrium, P would

differ sensibly from R, and the sum of the terms written down would bear a finite ratio to s. Otherwise expressed, the distinction between a position of equilibrium and any other position lies in the fact that the work done in a small displacement is of the second order (at least) instead of being of the first order according to the displacement. This distinction is summarily expressed by the zero on the right side of the equation.

As an equation of this type applies to each particle of the system, in the sum

$$\Sigma P \cos \phi \,.\, s + \Sigma R \cos \theta \,.\, s = 0,$$

where s is a quite arbitrary small displacement of the point of application of each force, and ϕ, θ are the angles which P and R make with s. The point of application must be a fixed point of the system, capable of identification in the course of the displacement. The displacements admitted may not be actual, but purely imaginary or *virtual*, and the corresponding work of the forces is *virtual work*.

204. The power of the principle of virtual work resides in the fact that the displacements are arbitrary and independent. Hence as many independent equations of the above type can be formed as the number of forces involved, and the forces in equilibrium can therefore be determined. Even the geometrical conditions can be violated, though it must be understood that such conditions take the form of constraints with corresponding forces, which then enter into the equations. For practical reasons it is clearly convenient to give the equations as simple a form as possible, and this can be done by restricting or isolating the types of displacement admitted. Thus by keeping the point of application of a force P stationary ($s=0$) this force is excluded from the equation. The same result follows when the point is given a displacement at right angles to P ($\phi=90°$). The distinction between external forces P and internal actions R becomes quite artificial when the constraints are violated. When the geometrical constraints are preserved, the virtual work done by the components of a stress will disappear, or not, according to the same considerations as those already noticed in dealing with the principle of energy. Thus the stresses in a rigid body or an inextensible string will not occur in the equation so long as the property of rigidity is maintained.

The virtual work done by a frictional force $\mu' R$ in a displacement s of the point of contact is $-\mu' R \,.\, s$, only so long as the sense of the displacement agrees with the actual tendency to motion.

The work done by a couple in a small rotation about a point O in its plane can be expressed very simply. Let A, B be the feet

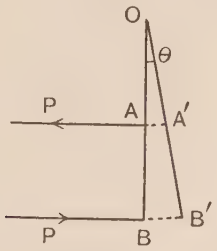

Fig. 119.

of the perpendiculars from O on the components P. In a small rotation A, B are displaced to A', B' at right angles to OAB, and $AA' = OA \,.\, \theta$, $BB' = OB \,.\, \theta$, where θ is the small rotation about O. Hence the work done by the couple is

$$P.\,BB' - P.\,AA' = P(OB \,.\, \theta - OA \,.\, \theta)$$
$$= P.\,AB \,.\, \theta = M\theta,$$

or the product of the moment of the couple and the rotation (in circular measure). It is independent of the position of the point O.

205. A statical machine is a system in which one force, usually a load under gravity, is maintained in equilibrium by the application of another force, differing from the first in magnitude or direction. When the relation between these two forces alone is required, it is most simply obtained by the principle of virtual work.

Ex. A wheel and axle consists of two connected circular cylinders possessing a common axis, about which they are free to rotate. Strings passing round them in opposite directions support the weights L and F.

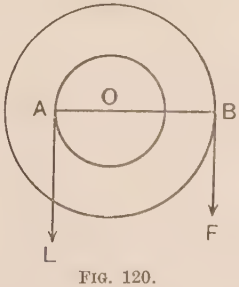

Fig. 120.

When the cylinders are rotated through an angle θ, the axle of radius a takes up a length $a\theta$ of the string and L is raised by this amount. At

the same time the wheel, of radius b, pays out the length $b\theta$ of string and F descends to this extent. In the operation the weights alone do work, and therefore when the system is in equilibrium

$$L \,.\, a\theta - F \,.\, b\theta = 0.$$

Thus $$F/L = a/b$$

is the required relation between F and L. The mechanism in this case being exceptionally simple, the same result can be obtained with equal facility by taking moments about the axis O.

206. *Ex.* To find the relation between F and L in the pulley system represented, in which all the strings are attached at one end to L and at the other to pulleys of weight $w_1, w_2, \ldots$ in succession. All the strings hang vertically, and the first passes over a fixed pulley, while the last is held in equilibrium by the force F. The number of moveable pulleys is n.

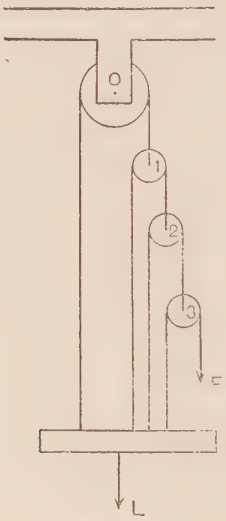

FIG. 121.

If L be drawn downwards a distance x, the moveable pulleys and F are all raised. Let the rth pulley be raised a distance u_r. The length of string passing over it is increased by $x+u_r$ on the left side and decreased by $u_{r+1}-u_r$ on the right side in the figure. Hence

$$x + u_r = u_{r+1} - u_r,$$

$$x = u_{r+1} - 2u_r.$$

The point of application of F takes the place of the $(n+1)$th pulley and is raised a distance y, where

$$x = y - 2u_n,$$

while for the first pulley $u_1 = x$ clearly. Thus the displacements are related by the equations :

$$\begin{aligned} u_{r+1} + x &= 2(u_r + x) = 2^2(u_{r-1} + x) \\ &= \ldots = 2^r(u_1 + x) \\ &= 2^{r+1}x. \end{aligned}$$

Hence $$u_{r+1} = (2^{r+1} - 1)x,$$

and finally $$y = (2^{n+1} - 1)x.$$

Now the equation of virtual work, in which the tensions of the strings play no part, is

$$Lx - w_1u_1 - w_2u_2 - \ldots - w_nu_n - Fy = 0.$$

Hence the condition for equilibrium is

$$F = \frac{L - w_1 - 3w_2 - \ldots - (2^n - 1)w_n}{2^{n+1} - 1}.$$

207. *Ex.* Four bars, freely jointed, form a quadrilateral. Forces from one point act at right angles to each bar, each being proportional to the length of the bar on which it acts. To prove that, for equilibrium, the quadrilateral is inscribable in a circle.

Were the quadrilateral rigid, equilibrium would follow, provided that the forces are directed in the sense of the sides taken in order and rotated through a right angle. But the quadrilateral is not rigid, and a special condition is necessary for equilibrium.

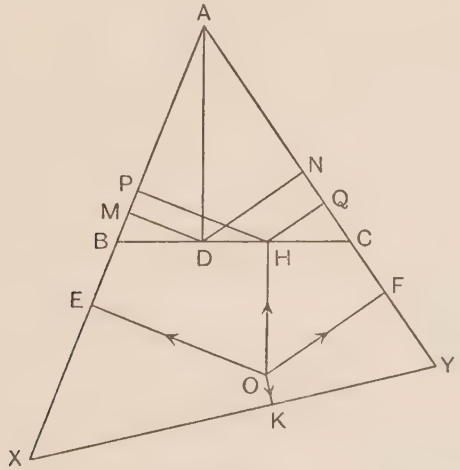

FIG. 122.

Let $BXYC$ be the quadrilateral, the sides BX, CY meeting in A. Let E, K, F, H be the feet of the perpendiculars on the sides from the point O, from which the forces are directed. Let D be the foot of the perpendicular from A on BC, and M, N, P, Q the feet of the perpen-

diculars from D, H on AB, AC. The angles of the triangle ABC may be denoted by A, B, C in the usual sense.

The points X, Y being kept fixed, the quadrilateral may be deformed by rotating the bar BC through a small angle α about A; this involves a rotation of XB through a small angle β and of CY through a small angle γ. From the displacement of B, which is at right angles to AX,

$$XB \cdot \beta = AB \cdot \alpha,$$

and from the displacement of C, which is at right angles to AY,

$$YC \cdot \gamma = AC \cdot \alpha.$$

The displacement of E is $XE \cdot \beta$ along OE, that of F is $YF \cdot \gamma$ along FO. The displacement of H is that relative to D, or $DH \cdot \alpha$ along HO, together with that of D, or $AD \cdot \alpha$ along CB and at right angles to OH. Hence the work done by the forces in the deformation is

$$XB \cdot XE \cdot \beta - CY \cdot YF \cdot \gamma - BC \cdot DH \cdot \alpha,$$

and therefore, for equilibrium,

$$AB \cdot XE - AC \cdot YF - BC \cdot DH = 0.$$

Thus
$$AB \cdot XE - AC \cdot YF = DH(AB \cos B + AC \cos C)$$
$$= AB \cdot PM + AC \cdot NQ,$$

or
$$AB(XE - PM) = AC(YF + NQ). \quad \ldots\ldots(1)$$

But
$$AM = AD \cdot \sin B, \quad AN = AD \cdot \sin C,$$
$$PE = OH \cdot \sin B, \quad QF = OH \cdot \sin C,$$

whence
$$AB(AM + PE) = AC(AN + QF). \quad \ldots\ldots(2)$$

By the addition of (1) and (2),

$$AB \cdot AX = AC \cdot AY,$$

which proves that $XBCY$ is inscribable in a circle.

208. *Ex.* Two blocks, separated symmetrically by a rough wedge of angle 2α, are impelled towards each other by equal forces X. To discuss the conditions of equilibrium when the blocks are capable of

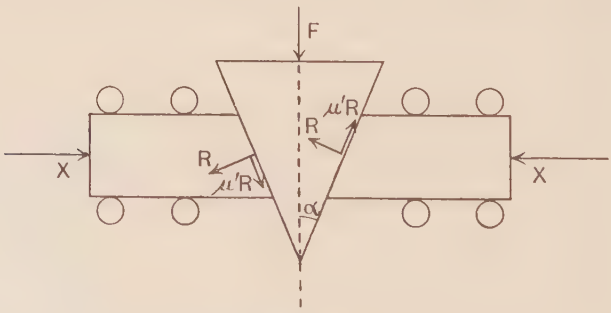

FIG. 123.

moving only in the direction of X and a perpendicular force F is applied to the base of the wedge.

Let R be the normal reaction over the bearings of the wedge and $\mu' = \tan\lambda'$ the coefficient of friction actually brought into play. The equilibrium of the wedge requires the condition

$$F = 2(\mu' R\cos\alpha + R\sin\alpha) = 2R\sin(\lambda' + \alpha)/\cos\lambda',$$

while the equilibrium of either block requires the condition

$$X = R\cos\alpha - \mu' R\sin\alpha = R\cos(\lambda' + \alpha)/\cos\lambda'.$$

To obtain equivalent conditions by the principle of virtual work, let the wedge be moved forward a distance y and the blocks be displaced outward a distance x. The surfaces of the wedge are then displaced $y\sin\alpha$ outward and $y\cos\alpha$ towards the vertex, while the bearings of the blocks are displaced $x\cos\alpha$ outward and $x\sin\alpha$ from the vertex of the wedge. These being the normal and tangential components of the displacements of the bearing surfaces, the equation of virtual work is

$$Fy - 2R(y\sin\alpha - x\cos\alpha) - 2\mu' R(y\cos\alpha + x\sin\alpha) - 2Xx = 0.$$

Here x and y are to be regarded as arbitrary and independent displacements, regardless of geometrical constraints, and this single equation contains the required solution. When $x = 0$, the first of the above equations results; when $y = 0$, the second equation is obtained. In the only displacement of the system which is actually possible, the blocks remain in contact with the wedge and

$$y\sin\alpha = x\cos\alpha,$$

so that

$$F\cos\alpha - 2\mu' R - 2X\sin\alpha = 0.$$

On the other hand, if

$$y\cos\alpha + x\sin\alpha = 0,$$

there is no lateral displacement at the bearings, but the wedge sinks into the blocks (in the virtual sense) to the depth $y\sin\alpha - x\cos\alpha$.

The equation of virtual work now gives

$$F\sin\alpha - 2R + 2X\cos\alpha = 0.$$

From the equations of equilibrium in their first form,

$$F\cos(\lambda' + \alpha) = 2X\sin(\lambda' + \alpha).$$

If λ is the angle of friction, the wedge is on the point of moving forward when $\lambda' = \lambda$. If $\lambda + \alpha > 90°$, $\lambda' = 90° - \alpha$ is a possible value, and then no force F, however great, can overcome the forces X, however small. The condition of limiting friction is only reached when

$$\lambda + \alpha < 90°,$$

and then

$$F/2X = \tan(\lambda + \alpha).$$

If α and λ are small, this ratio is small, and its reciprocal is a measure of the "mechanical advantage" of the arrangement in separating the blocks against given forces X. In the absence of friction ($\lambda = 0$) the corresponding advantage would be $\cot\alpha$, and therefore the "efficiency" of the actual arrangement compared with this standard is

$$\cot(\lambda + \alpha)/\cot\alpha = \tan\alpha\cot(\lambda + \alpha).$$

On the other hand, the wedge is on the point of being forced out if $\lambda' = -\lambda$. If $\lambda > \alpha$, $\lambda' = -\alpha$ is a possible value, and then no forces X, however great, can overcome a force F, however small. The condition of limiting friction in this direction is only reached when $\lambda < \alpha$, and then

$$F/2X = \tan(\alpha - \lambda).$$

This ratio, which is large only when the acute angle α is large and λ is small, is a measure of the mechanical advantage of the arrangement in moving the wedge against a given resistance F. The frictionless standard would give $\tan\alpha$ for the corresponding advantage, and therefore the efficiency of the actual arrangement is

$$\tan(\alpha - \lambda)/\tan\alpha = \cot\alpha \tan(\alpha - \lambda).$$

This example illustrates the distinction which must be drawn in the use of the terms advantage and efficiency in relation to a given mechanical arrangement, according to the purpose for which it is intended.

209. With regard to the equation of virtual work,

$$\Sigma P \cos\phi \,.\, s + \Sigma R \cos\theta \,.\, s = 0,$$

it has been explained that when s stands for a small finite displacement from the position of equilibrium, and the forces P and R vary accordingly, the work done in the displacement is not actually zero, as represented, but at least of the second order in s.

The general equation of work and energy (§ 129) has been written in the form

$$\Sigma \tfrac{1}{2}mV^2 - \Sigma \tfrac{1}{2}mU^2 = \Sigma \int P \cos\phi \,.\, ds + \Sigma \int R \cos\theta \,.\, ds.$$

When the displacements are reckoned from a position of equilibrium, and are very small, this equation may be written

$$\Sigma \tfrac{1}{2}mv^2 - \Sigma \tfrac{1}{2}mu^2 = \Sigma P \cos\phi \,.\, s + \Sigma R \cos\theta \,.\, s,$$

and both sides of the equation are of the second order (not the first) in the displacements. Here $\Sigma \frac{1}{2}mu^2$ is the kinetic energy of the system when passing through the position of equilibrium. The displacements being now actual ones, and the forces conservative (so that sliding friction is excluded),

$$\delta E = \delta W,$$

where δE is the increase of kinetic energy and δW, the work done, depends only on the initial (or equilibrium) and final (or disturbed) conditions of the system. Since potential energy Q is the reverse of work done in passing from a standard position,

$$\delta E = -\delta Q.$$

This result follows directly from the fact (§ 130) that the sum of the kinetic and potential energies in a conservative system is constant. While, however, the variations in the two forms of energy are in general of the first order in terms of the displacements, it is now seen that when the displacements represent a small disturbance from a position of equilibrium, the variations are at least of the second order.

210. From the equation

$$\delta E = \Sigma \tfrac{1}{2} m v^2 - \Sigma \tfrac{1}{2} m u^2 = -\delta Q$$

important deductions can immediately be drawn.

(*a*) If the potential energy Q is constant, $\delta Q = 0$ exactly. If the system in equilibrium be given a small kinetic energy, that energy will remain constant, with no tendency to increase or disappear. Or, if the system be displaced to a neighbouring position without motion ($\Sigma \frac{1}{2} m v^2 = 0$), the system will remain stationary. In these circumstances the equilibrium is said to be *neutral*.

A uniform sphere on a horizontal plane satisfies these conditions and is in neutral equilibrium. In the ordinary statical machine, such as a pulley system, without friction, the effective conditions are unaltered by a displacement and equilibrium is of the neutral type.

(*b*) If in any position of the system the potential energy Q is an absolute minimum, δQ is positive for all displacements. Then $\Sigma \frac{1}{2} m v^2 < \Sigma \frac{1}{2} m u^2$. If the system in equilibrium suffers a disturbance in the form of a small amount of kinetic energy, that energy has no tendency to increase. This implies that the departure from the condition of equilibrium is also strictly limited, large changes in kinetic or potential energy being precluded.

This type of equilibrium is called *stable*. It is illustrated by a particle at the lowest point of a smooth concave bowl.

(*c*) If in any position of the system the potential energy Q is an absolute maximum, δQ is negative for all displacements. Then $\Sigma \frac{1}{2} m v^2 > \Sigma \frac{1}{2} m u^2$, and kinetic energy, however small in the position of equilibrium, tends to increase. If the system be displaced to a neighbouring position without motion ($\Sigma \frac{1}{2} m v^2 = 0$) it cannot return to the position of equilibrium, for this would imply $\Sigma \frac{1}{2} m u^2$ negative, which is impossible. The departure tends to increase, therefore, and the type of equilibrium is *unstable*. It is illustrated by a particle resting at the highest point of a smooth convex surface.

(*d*) It may happen that the potential energy Q is a minimum (δQ positive) for some displacements and a maximum (δQ negative) for others. This implies a combination of the previous circumstances according to the type of the displacement. Since the element of instability is present, this will assert itself and the type of equilibrium must be regarded as unstable.

This condition is illustrated by a particle placed at the level point on a saddle-shaped surface. The position is one of equilibrium, but it is unstable.

211. It is now clear that positions in which the potential energy of the system is a minimum or a maximum, including those which unite both conditions according to the type of displacement, are positions of equilibrium. Further than this, when a displacement of the system does not leave the potential energy unchanged, there are no others. In general, in the course of a small continuous displacement of the system in one direction the potential energy either increases or decreases. The change depends on the first order of the displacement and is reversed in sign with the direction of the displacement. When the displacement is made from the position of equilibrium, the change is of the second order and its sign is independent of the direction of the displacement. This implies that in the position of equilibrium the potential energy reaches a turning value, so that from increasing it begins to decrease, or from decreasing it begins to increase. But these are the conditions which define a maximum or minimum value, though these conditions may coexist, as already seen. Thus the problem of finding the positions of equilibrium of a system under conservative forces is reduced to the problem of finding the positions in which the potential energy is a maximum or a minimum. The equilibrium is stable only when the potential energy is an absolute minimum.

When the external forces are those due to gravity alone, the potential energy is (§ 131)

$$Q = g \,.\, \Sigma mh = Mg \,.\, \bar{h},$$

where M is the mass of the system and $\bar{h}$ is the height of its centre of gravity above a fixed horizontal plane. Hence the positions of equilibrium are those in which this height is a minimum or a maximum, and stability is associated with the minimum condition alone. When the system is displaced in all possible ways the centre of gravity will in general trace out a surface. Those points of this

surface which are at a lower level than all the points in their immediate neighbourhood define positions of stable equilibrium. All positions of equilibrium, stable or unstable, are defined by those points on the locus of the centre of gravity where the surface is level, that is, where the tangent plane is horizontal.

212. *Ex.* A uniform square board is supported in a vertical plane with two of its sides resting on two small smooth pegs, a distance d apart, at the same horizontal level. To discuss the positions of equilibrium.

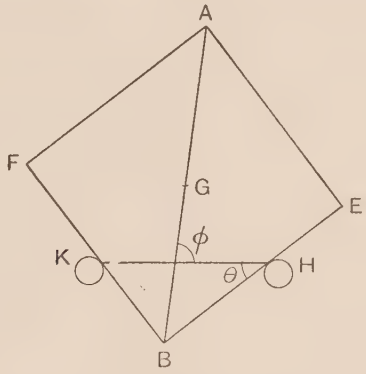

FIG. 124.

Let a be a side of the square $AEBF$ and $D = a\sqrt{2}$ its diagonal. The height h of the centre G above the level of the pegs H, K, being the mean of the heights of the points E, F, is given by

$$\begin{aligned} 2h &= (a - d\cos\theta)\sin\theta + (a - d\sin\theta)\cos\theta \\ &= a(\sin\theta + \cos\theta) - d\sin 2\theta, \end{aligned}$$

where θ is the angle KHB. If $\phi = \theta + 45^\circ$ is the inclination of AB to the horizon,

$$2h = D\sin\phi + d\cos 2\phi.$$

When the board is slightly disturbed ϕ becomes $\phi + \epsilon$ and h becomes h', where

$$\begin{aligned} 2h' &= D\sin(\phi+\epsilon) + d\cos 2(\phi+\epsilon) \\ &= D(\sin\phi\cos\epsilon + \cos\phi\sin\epsilon) + d(\cos 2\phi\cos 2\epsilon - \sin 2\phi\sin 2\epsilon) \\ &= D\sin\phi(1 - \tfrac{1}{2}\epsilon^2) + D\cos\phi\,.\,\epsilon + d\cos 2\phi(1 - 2\epsilon^2) - d\sin 2\phi\,.\,2\epsilon \end{aligned}$$

when higher powers of ϵ are neglected. Hence

$$2(h - h') = \epsilon(2d\sin 2\phi - D\cos\phi) + \tfrac{1}{2}\epsilon^2(D\sin\phi + 4d\cos 2\phi).$$

The positions of equilibrium are now defined by the vanishing of the coefficient of the first power of ϵ, or

$$\cos\phi(4d\sin\phi - D) = 0.$$

The first position, given by $\cos\phi = 0$, $\phi = 90°$, is the symmetrical position in which the diagonal AB is vertical. Here

$$4(h - h') = \epsilon^2(D - 4d),$$

and this shows that $h < h'$ and h is a minimum, if $d > \frac{1}{4}D$. The equilibrium is then stable. If $d < \frac{1}{4}D$, $h > h'$, h is a maximum and equilibrium is unstable.

Other positions of equilibrium, on either side of the position of symmetry, are given by

$$\sin\phi = D/4d.$$

These exist only when $d > \frac{1}{4}D$, or the symmetrical position is one of stable equilibrium. In this case

$$\begin{aligned} 4(h - h') &= \epsilon^2(D\sin\phi + 4d\cos 2\phi) \\ &= \epsilon^2 \cdot 4d(\sin^2\phi + \cos 2\phi) = \epsilon^2 \cdot 4d\cos^2\phi, \end{aligned}$$

and is certainly positive. Thus $h\,(> h')$ is a maximum, and the two unsymmetrical positions of equilibrium, if they exist, are unstable. There is, however, another condition necessary for their existence. For, when the two sides of the board are in contact with the pegs, it is evident that $\phi > 45°$. Hence $4d/D < \sqrt{2}$ or $d < \frac{1}{2}a$.

Care must always be taken to see that an analytical result is actually possible. Thus, in this instance, it is not enough that the (unlimited) line FB touches the peg K, but it is also necessary that the point of contact falls in the limited section FB of that line.

213. When one body is at rest under gravity in contact with the rough surface of another, which is fixed, the nature of the equilibrium is decided by similar principles when the possibility of slipping is

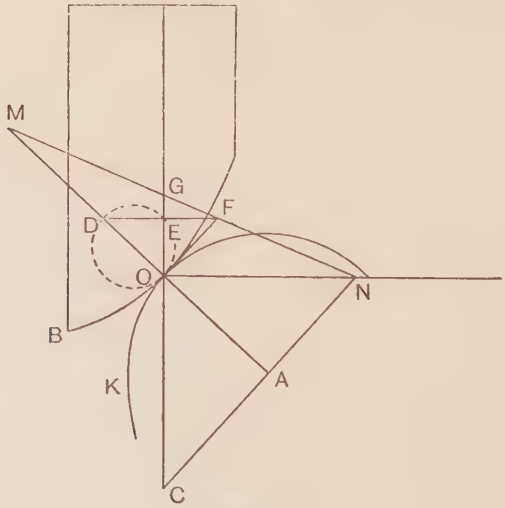

FIG. 125.

excluded, so that the point of contact is the instantaneous centre of a small displacement. The fixed surface is the fixed centrode in this displacement, the surface of the supported body is the moving centrode, and stability depends on whether the path of the centre of gravity of this body is concave in the upward or downward direction.

Let MOA be the common normal to the two surfaces in contact at O, $OA=R$ being the radius of curvature of the fixed surface OK and $OM=\rho$ being the radius of curvature of the supported surface OB. Since the action between the two surfaces is located at O, the weight of the supported body must also pass through O, or, if G is the centre of gravity, the line GO is vertical. This is the condition of equilibrium; O being the instantaneous centre, the initial displacement of G is horizontal. Let α denote the acute angle MOG, or the inclination of the two surfaces to the horizon at their point of contact.

Let MG cut the horizontal line through O in N, and let NA cut the vertical line GO in C. By this construction the point C is the centre of curvature of the path of G when the arc OB begins to roll on the arc OK (cf. § 26). In this figure C is below G, the height of G is a maximum and the position is one of unstable equilibrium.

Draw OF parallel to CN to meet MN in F, and FED parallel to ON to meet OG in E and OM in D. Denote GO by h and EO by k.

Then (§§ 25-28), if $\sigma=GC$ be the radius of curvature,

$$\frac{1}{DO}=\frac{1}{R}+\frac{1}{\rho},$$

$$\frac{1}{k}=\sec\alpha\left(\frac{1}{R}+\frac{1}{\rho}\right)$$

$$=\frac{1}{h}+\frac{1}{\sigma-h},$$

whence

$$\sigma-h=\frac{hk}{h-k},\quad \sigma=\frac{h^2}{h-k},$$

or $GC=h^2/GE$. This relation, when interpreted with due regard to the signs of the sections GC, GE, implies that the points E and C both lie on the same side of G. Hence the path of G is concave downwards, and equilibrium is unstable, if G is vertically above the point E. If G is vertically below E, or $h<k$, the position of equilibrium is stable.

In the figure the surfaces in contact are convex to each other, and the radii of curvature R, ρ are taken to be positive. Either of

the convexities, but not both, may be reversed, and the sign of R or ρ must be changed accordingly. If R is negative, $-R>\rho$, and if ρ is negative, $-\rho>R$; these conditions follow simply from the fact that the supporting surface must lie below the supported surface. In either case $R+\rho$ and $R\rho$ are negative, and k remains positive. Thus in all cases E is a point vertically above the point of contact O. The shape of the supported body being undefined except at the point of contact, its centre of gravity G may be vertically underneath O. In this case, illustrated by a familiar class of toys, equilibrium is always stable.

214. When the intermediate condition, in which the points G and E coincide, is presented, so that

$$\frac{1}{h}=\frac{1}{k}=\sec\alpha\left(\frac{1}{R}+\frac{1}{\rho}\right),$$

discrimination between stability and instability is not quite so simple.

The centre of gravity now lies on the circle described on OD as diameter, or the circle of inflexions (§ 30). In general this defines the nature of the path of G when displaced, and equilibrium is stable or unstable according to the direction of the displacement, or on the whole unstable. But this conclusion is not assured. The centre of curvature C is now at infinity and the initial path of G is straight to a higher order of approximation than before, but whether the path possesses a point of inflexion or a higher type of singularity is a question requiring further examination. This needs a development to a higher order of approximation and involves a knowledge of the curves in contact beyond that represented by the simple curvatures.

Ex. A sphere of radius a rests on a sphere of radius b, the line of centres being vertical ($\alpha=0$), with the centre of gravity in the critical position given by

$$\frac{1}{h}=\frac{1}{k}=\frac{1}{a}+\frac{1}{b}.$$

To find the condition of stability.

Let $a\phi=b\theta$ be equal corresponding arcs on the moving and fixed spherical centrodes, centres A and B. If h' is the height of G above the initial point of contact O in the displaced position, since $AG=a-h$,

$$h'=-b+(a+b)\cos\theta-(a-h)\cos(\theta+\phi),$$

$$h'-h=(a-h)\{1-\cos(\theta+\phi)\}-(a+b)(1-\cos\theta)$$

$$=(a-h)\{\tfrac{1}{2}(\theta+\phi)^2-\tfrac{1}{24}(\theta+\phi)^4+\ldots\}-(a+b)(\tfrac{1}{2}\theta^2-\tfrac{1}{24}\theta^4+\ldots).$$

But

$$a-h=\frac{a^2}{a+b},\quad \theta+\phi=\frac{a+b}{a}\theta.$$

Hence

$$2(h'-h)=\frac{a^2}{a+b}\left\{\theta^2\left(1+\frac{b}{a}\right)^2-\tfrac{1}{12}\theta^4\left(1+\frac{b}{a}\right)^4+\dots\right\}-(a+b)(\theta^2-\tfrac{1}{12}\theta^4+\dots)$$

$$=-\tfrac{1}{12}\theta^4(a+b)\left\{\left(1+\frac{b}{a}\right)^2-1\right\}+\dots$$

$$=-\tfrac{1}{12}\theta^4\cdot\frac{b}{a^2}(a+b)(2a+b)+\dots\,.$$

For stability, which is thus decided by terms of the fourth order, $h'>h$ and either a or b is negative. But if a is negative, $-a>b$ and

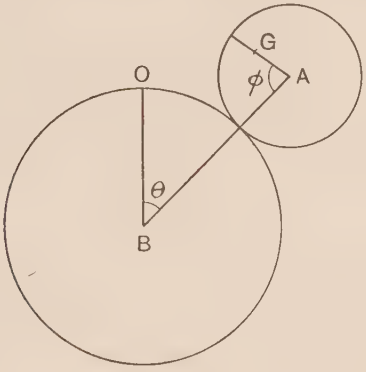

FIG. 126.

$h'<h$. Hence equilibrium on the upper surface of a fixed sphere is unstable in the given conditions. If b is negative, $a+b$ is also negative and for stability $2a+b$ must be negative. Hence the sphere can only rest in stable equilibrium on the lower concave surface of a sphere of more than double its radius. If $-b=2a$, $\phi=-2\theta$, $2a=h=h'$, G moves horizontally when displaced, and equilibrium is neutral. In all other circumstances equilibrium in this critical condition is unstable.

EXAMPLES.

1. Forces are represented completely by AB, BC, CD, DE, where ABC, ACD, ADE are right angles in the same direction, and $AB=104$, $BC=153$, $CD=672$ and $DE=696$. Prove that the resultant is 985, and find the position of its line of action.

2. A coplanar system of forces has equal moments about three points not in the same straight line. Show that it is equivalent to a couple.

3. The moment of a system of coplanar forces is equal about the points A, B, and also about the points C, D. Show that either AB, CD are parallel or the system is equivalent to a couple.

4. The moments of a coplanar system of forces about the points A, B, C are M_1, M_2, M_3 respectively. Prove that the moment of the system about any point O in the plane is $M_1x+M_2y+M_3z$, where x, y, z are the ratios of the triangles OBC, OCA, OAB to ABC.

5. A, B, C, D are four concyclic points in the order given. Forces act along AB, BD, DC, CA in the sense indicated and are in equilibrium. Prove that they are respectively proportional to CD, AC, AB, BD.

6. Forces represented by AB, BC, CA act at the points A, B, C. If they are all turned about these points through a given angle in the same direction, show that they will form a couple. Prove that the greatest moment this couple can have is represented by

$$(b^2c^2+c^2a^2+a^2b^2)^{\frac{1}{2}},$$

a, b, c being the sides of the triangle ABC.

7. Forces P, Q, R act along the sides of a triangle, the lengths of which are a, b, c. If the resultant acts on the line joining the centre of gravity of the triangle to the centre of the inscribed circle, prove that

$$\frac{P}{a(b-c)}=\frac{Q}{b(c-a)}=\frac{R}{c(a-b)}.$$

8. In a system of coplanar forces the typical force has components (X, Y) acting at the point (x, y). The rigid body on which the forces act remains in equilibrium when the forces are turned in the plane about their points of application through any assigned angle. Prove that this is possible if the conditions

$$\Sigma X=\Sigma Y=0,$$
$$\Sigma(xY-yX)=\Sigma(xX+yY)=0$$

are satisfied.

9. A system of coplanar forces with components (X, Y) acts on a rigid body at points (x, y). Prove that when the forces are rotated together through any given angle about their points of application, the line of their resultant R passes through a fixed point (h, k), where

$$hR^2=\Sigma X\,.\,\Sigma(xX+yY)+\Sigma Y\,.\,\Sigma(xY-yX),$$
$$kR^2=\Sigma Y\,.\,\Sigma(xX+yY)-\Sigma X\,.\,\Sigma(xY-yX).$$

10. A smooth shallow bowl in the form of a segment of a sphere stands with its rim horizontal. A uniform rod is placed in a vertical plane through the centre, with one end resting inside the bowl and the other projecting beyond the rim. Prove that the lower end must rest vertically below some point on the diameter of the rim through the other point of support, which is situated in the third of the diameter more remote from the point of contact with the rim.

11. A light string of length l is suspended from two points A and B on the same level a distance a apart. A smooth heavy ring R of weight W slides on the string and is supported in an unsymmetrical position by

a horizontal force P acting on the ring. Prove that the horizontal displacement of the ring is

$$\tfrac{1}{2}l^2P\{l^2P^2+(l^2-a^2)W^2\}^{-\frac{1}{2}}.$$

12. The ends of two strings, whose lengths are b and d, are tied to two pegs at a distance a apart in a horizontal line. The other two ends are tied to the extremities of a uniform bar whose length is c. Prove that when the system is at rest and the directions of the strings are at right angles, the ratio of the tension of the strings is

$$(ab+cd)/(ad+bc).$$

13. A light girder is supported at its two ends in a horizontal position. A trolley suspended by two wheels bearing an equal load runs along it. Prove that the bending moment at any given point of the girder is greatest when one wheel is over the point and the other wheel is further from the centre of the girder. Prove also that the greatest bending moment along the girder occurs at those points whose distances from the centre of the girder are equal to a quarter of the distance between the wheels.

14. A uniform heavy beam is supported symmetrically in a horizontal position on two props. Prove that when the greatest bending moment at any point of the beam is as small as possible, the fraction of the whole length of the beam between the two props is $2-\sqrt{2}$.

15. Weights W_1, W_2, ... are attached at successive points of a light string tied at its ends to fixed points. The inclinations to the horizon of the two sections of the string on either side of W_r are ψ_{r-1}, ψ_r. Show that

$$W_r=c(\tan\psi_r-\tan\psi_{r-1}),$$

where c is constant. Hence prove (a) that along a uniform chain suspended by its ends the radius of curvature varies as $\sec^2\psi$, where ψ is the slope of the chain at the point, (b) that if a chain suspended by its ends assumes the form of a circular arc, the density at any point varies inversely as the square of the depth of the point below the centre of the circle.

16. A pentagon formed of five heavy equal uniform jointed bars is suspended from one corner, and the opposite side is supported by a string attached to its middle point of such length as to make the pentagon regular. Prove that the tension of the string is $4W\cos^2 18°$, and that the thrust in the lower bar is $\frac{1}{4}W\sec 18°$, where W is the weight of any rod.

17. A regular hexagon composed of freely jointed bars, each of weight w, is supported at one vertex. A weight W is hung from the opposite lower vertex, and the hexagon is prevented from collapsing by a light horizontal bar in contact with the two vertical sides. Find the position of this bar, and the stress along it.

18. Two weights stand on a rough plane, being connected by a string fully stretched. The coefficient of friction for one weight is $\frac{5}{8}$, for the other, double the weight of the first, $\frac{1}{3}$. Prove that it is possible to rock the plane about a line at right angles to the string through the angle 45° without disturbing the equilibrium of the weights.

19. A uniform rod is attached by a string of twice its length tied at one end to a point on a rough vertical wall. The other end of the rod rests in contact with the wall at a point vertically below the point of suspension. If $\mu(<\sqrt{3})$ is the coefficient of friction between the rod and the wall, prove that the greatest inclination θ of the string to the wall is given by

$$\tan\theta = 2\mu(3+\mu^2)^{-1},$$

and discuss the range of possible positions of the rod.

20. A uniform rod rests inside a rough sphere in a vertical plane through the centre. Prove that in the limiting position of equilibrium the inclination θ of the rod to the horizon is given by

$$\cot\theta = (\cos 2\alpha + \cos 2\lambda)/\sin 2\lambda,$$

where λ is the angle of friction and 2α is the angle which the rod subtends at the centre of the sphere.

21. Two equal uniform bars rigidly joined to form an angle 2α are placed in a vertical plane astride a rough cylinder with its axis horizontal. The points of contact are the middle points of the rods, and λ is the angle of friction. If θ is the inclination of the internal bisector of the angle between the rods to the vertical in the limiting position, prove that

$$\cot\theta = (\cos 2\lambda - \cos 2\alpha)/\sin 2\lambda.$$

22. A cylinder rests on the ground with its axis parallel to a vertical wall. A second equal cylinder rests in contact with the first and with the wall, so that the plane containing the two axes makes the angle 2α with the wall. Show that the coefficient of friction between the wall and the upper cylinder must be greater than unity, that the angle of friction between the two cylinders is greater than α, and that

$$\tan\alpha < 2\mu(1-\mu)^{-1}$$

if μ is the coefficient of friction between the lower cylinder and the ground.

23. Two rough cylinders of equal radius rest with their axes horizontal in contact with each other on a plane inclined at an angle α to the horizon. If W_1 is the weight of the upper cylinder and W_2 of the lower, show that $W_1 > W_2$ and that the coefficient of friction between the two cylinders is greater than $(W_1+W_2)/(W_1-W_2)$. If μ is the coefficient of friction between the plane and the upper cylinder, prove that

$$\tan\alpha < \frac{2\mu}{\mu+1}\cdot\frac{W_1}{W_1+W_2}.$$

24. Deduce from the principle of virtual work the conditions of equilibrium of a coplanar system of forces acting on a rigid body in the forms (*a*) that the sum of the components in any direction vanishes, (*b*) that the sum of the moments about any point vanishes.

25. PQ, QR, RS, SP are four equal uniform bars, each 1 lb. in weight. They are freely jointed so as to form a square, PQ being fixed in a vertical position with P uppermost. Equilibrium is maintained by an inextensible string joining R to the middle point of SP. Calculate the tension of the string.

26. Four equal uniform rods form a rhombus, freely jointed. The frame rests in equilibrium with one diagonal vertical and the two lower rods supported on two smooth pegs at the same horizontal level. If a is the length of each rod, $2c$ the distance between the pegs and $2b$ the horizontal diagonal of the frame in the position of equilibrium, prove that $b^3 = a^2 c$.

Show that there is no reaction at the bottom point of the frame when $8a = 5\sqrt{5} \cdot c$.

27. The rods AB, BE, EF, FO are freely jointed together and form a linkage in a vertical plane. The middle point M of AB is fixed by a smooth pivot, and the end of the rod FO is similarly fixed at a point O below AB. In the normal position AB, EF are horizontal, and BE vertical. Another vertical rod CD connects by smooth joints a point C in MB to a point D in OF.

Prove that a weight W placed at any point on the rod EF will be in equilibrium with an equal weight applied at A, provided

$$OD/OF = MC/MB.$$

28. A light string passing over two smooth pulleys in a horizontal line has two equal weights W_2 attached to its ends. A weight W_1 is attached to the string midway between the pulleys. When the system is in equilibrium the angle between the two parts of the string at W_1 is 4α, but when W_1 is released from the point midway between the pulleys it comes to rest again with the parts of the string making the angle 4β with one another. Prove that $\tan\beta = \tan^2\alpha$.

29. A uniform rod AB of weight W and length a is freely pivoted at A. To the end B is tied a light string passing over a smooth pulley at C on the same level as A and carrying a weight W'. In the position of equilibrium the string is at right angles to the rod. When the rod is held in the horizontal position and released it comes to rest in the vertical position. Prove that $W' = \frac{3}{5}W$ and $AC = \frac{4}{3}a$.

30. A uniform square board of mass M is supported in a vertical plane on two smooth pegs at the same horizontal level. The distance between the pegs is a and the diagonal of the square is $D > 4a$. If one diagonal is vertical and a mass m is attached to its lower end, prove that the equilibrium is stable if

$$4am > M(D - 4a).$$

31. A uniform plate in the form of an equilateral triangle is placed in a vertical plane with two of its sides in contact with two smooth pegs on the same horizontal level. The side of the plate is a and the distance between the pegs is c. Prove that the position of equilibrium with one side horizontal is stable unless $a \geqslant 4c$, and that if $2c < a < 4c$, an unstable position of equilibrium exists in which $\cos^{-1} a/4c$ is the inclination of the third side to the horizon.

32. A uniform right cone stands with the centre of its base resting on the highest point of a sphere of radius R. Show that the equilibrium is stable if the height of the cone $h < 4R$. When $h = 4R$, prove that equilibrium is unstable.

PART IV

CHAPTER VIII

DYNAMICS OF A RIGID BODY IN TWO DIMENSIONS

215. When a rigid body is free to move, and the forces applied to it are no longer in statical equilibrium, every point of the body moves in its own particular way. Theoretically it should be possible to sub-divide the body into an indefinitely large number of very small parts approximating to the ideal particles considered at an earlier stage, to calculate the forces acting on each and to study the resulting individual motions in detail. This treatment would appear to be infinitely laborious, and it is in fact unnecessary. The general problem can be simplified in such a way that this complete analysis is superseded by more synthetic methods.

In the first place, from the kinematical point of view, the motion of the body as a whole can be specified in comparatively simple terms. A rigid body moving in space of three dimensions possesses six degrees of freedom, and its situation can be completely defined by six coordinates. When these are all expressed in terms of the time, the problem of the motion of the body is solved. The motion of any particular point of the body then becomes a question of pure geometry. But in this chapter the motion to be considered will be restricted to two dimensions, and the conditions are far simpler. This limitation implies that every point of the body is moving parallel to some plane fixed in space. Thus if any plane section of the body be taken parallel to this fixed plane, this section will move in its own plane and will carry the body with it. Now the motion of the section in its plane is defined by three coordinates. They may be taken to be the two coordinates which define the position of

a chosen point A in the section and the orientation of a line AB fixed in the body and lying in the same section. When the three coordinates are expressed in terms of the time the motion of the section, and of the body with it, is completely determined.

In the second place, from the dynamical point of view, the external forces act in general on the body at a certain number of points. The most familiar force, gravity, acts at all points. These external forces are transmitted through the body in the form of stresses which are completely modified when motion is in progress. To specify the internal forces at all points at every stage of the motion would be an extremely difficult task. It is a fortunate circumstance, to be seen shortly, that in finding the motion of the body as a whole the internal stresses can be ignored. It may be that the stresses must afterwards be calculated in order to secure a certain strength of construction according to the nature of the material employed, as in many engineering problems, but this process will be so much the simpler since the motion can first be calculated independently.

216. Consider the body, of total mass M, as divided into an indefinitely large number of small particles. At a given instant of

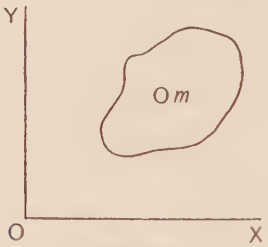

FIG. 127.

time the typical particle, of mass m, has the coordinates x, y. Its motion is defined by the equations

$$m\ddot{x} = X + R_x,$$
$$m\ddot{y} = Y + R_y.$$

Here $\ddot{x}$, $\ddot{y}$ are the components of the acceleration of m parallel to the two axes taken through a fixed point O, X and Y are the components of the external force applied to m, and R_x, R_y are the components of the internal reactions operating on m. From these equations may be deduced a third, namely,

$$m(x\ddot{y} - y\ddot{x}) = xY - yX + xR_y - yR_x,$$

where, on the right-hand side, stand the moments of the forces about the fixed point O. Now the force of which the components are R_x, R_y is the resultant of all the various forces which all the neighbouring particles exercise on m. To these Newton's law of Action and Reaction applies in detail, so that in every case the particle m, for every force to which it is subject, exerts a force which is equal, opposite and acting in the same straight line. Hence if all the corresponding equations of the above form are added together for the aggregate of particles which constitute the rigid body,

$$\Sigma R_x = \Sigma R_y = \Sigma(xR_y - yR_x) = 0\,;$$

and it then follows that

$$\Sigma m\ddot{x} = \Sigma X, \quad \text{.....(1)}$$

$$\Sigma m\ddot{y} = \Sigma Y, \quad \text{.....(2)}$$

$$\Sigma m(x\ddot{y} - y\ddot{x}) = \Sigma(xY - yX) = \mu, \quad \text{.....(3)}$$

where μ is the moment of all the external forces acting on the body about the fixed point O.

217. The product $m\ddot{x}$ (or $m\ddot{y}$), without regard to the actual forces from which it arises, is sometimes called the *effective force* acting on m parallel to the axis. It will now be seen that the equations just written down express the conditions that the aggregate of the external forces, together with the aggregate of the effective forces reversed, form a system in statical equilibrium. This is the statement of what is known as d'Alembert's Principle. Conversely, this principle, which involves no reference to the internal forces, suffices for writing down the same equations. But it will be evident that the principle adds nothing to the law of action and reaction, of which it is merely an alternative expression, directly deducible. The law itself must be regarded as the outcome of experience.

The simplest example of the idea represented by effective force is to be seen in centrifugal force (cf. § 114). For a particle of mass m rotating with angular velocity n at the distance a from a fixed centre it is equally possible to admit the acceleration an^2 towards the centre, or alternatively to add the centrifugal (or reversed effective) force man^2 in the opposite (outward) direction to the impressed forces acting. The latter course illustrates d'Alembert's Principle.

Let $\bar{x}$, $\bar{y}$ be the coordinates of the centre of gravity of the body so that

$$M\bar{x} = \Sigma mx, \quad M\bar{y} = \Sigma my, \quad M = \Sigma m.$$

Since quantities always equal must have the same rates of change,

$$M\dot{\bar{x}}=\Sigma m\dot{x}, \quad M\dot{\bar{y}}=\Sigma m\dot{y},$$

where $\dot{x}$, $\dot{y}$ are the components of the velocity of the centre of gravity. These are equivalent expressions of the linear momentum of the body parallel to the axes. In turn, their rates of change give

$$M\ddot{\bar{x}}=\Sigma m\ddot{x}, \quad M\ddot{\bar{y}}=\Sigma m\ddot{y},$$

where $\ddot{\bar{x}}$, $\ddot{\bar{y}}$ are the components of the acceleration of the centre of gravity.

Hence (1) and (2) become

$$M\ddot{\bar{x}}=\Sigma X, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$M\ddot{\bar{y}}=\Sigma Y, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

and these are precisely the equations which would result if the whole mass M were concentrated as a particle at the centre of gravity of the body, and the external forces X, Y were transferred with their original directions to the same point.

Equations (4) and (5) represent a generalization of Newton's second law (cf. § 115), and show that the motion of the centre of gravity of a rigid body is determined in a similar way to that of a particle, independently of any motion of rotation the body may possess about its centre of gravity. The latter motion remains to be considered.

218. In the simple type of motion which is the first to be discussed the body is constrained to rotate about a fixed axis. This axis is taken to pass through O at right angles to the plane Oxy. The origin O is thus a fixed point in space and in the body, every particle m describes a circle about the axis in a plane parallel to Oxy, and the condition of two-dimensional motion is clearly satisfied.

Let (r, θ) be the polar coordinates of m in the plane Oxy, so that

$$x=r\cos\theta, \quad y=r\sin\theta,$$

r being the perpendicular from m on the axis of rotation and θ the inclination of this perpendicular (or its projection on the plane) to Ox. Then

$$\begin{aligned} x\ddot{y}-y\ddot{x} &= r(\ddot{y}\cos\theta-\ddot{x}\sin\theta) \\ &= r\,.\,r\ddot{\theta}=r^2\ddot{\theta}, \end{aligned}$$

for in the circular motion of m the acceleration at right angles to the radius r is $r\ddot{\theta}$, and this can also be obtained by resolving the components $\ddot{x}$, $\ddot{y}$ in the direction making the angle $\theta+90^\circ$ with Ox.

The angular acceleration $\ddot{\theta}$ being the same for all the particles of the body, equation (3) now becomes

$$\ddot{\theta} \cdot \Sigma mr^2 = \Sigma(xY - yX) = \mu. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

This equation determines $\ddot{\theta}$, and from it the motion of the body about its fixed axis can be found, for the body in this case has only one degree of freedom.

Though one equation suffices to determine the motion when given forces are applied to the body, the fixing of the axis requires the application of forces which are not given but can be deduced by means of equations (4) and (5). For these forces on the axis enter into ΣX, ΣY, but not into the moment μ.

219. If, however, there is no fixed axis to constrain the motion, and yet the two-dimensional type of motion is assured, generally

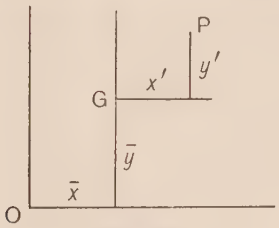

FIG. 128.

by considerations of symmetry, the motion of the centre of gravity is determined by (4) and (5), and it remains to adapt equation (3) so as to determine the rotation about this point.

Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of gravity G, (x, y) the coordinates of the point P of the body, referred to axes through the fixed point O, and (x', y') the coordinates of P referred to parallel axes through G. Then

$$x = \bar{x} + x', \qquad \Sigma mx' = 0,$$
$$y = \bar{y} + y', \qquad \Sigma my' = 0,$$

since, for the aggregate of particles such as m at P, $\Sigma mx = \bar{x}\Sigma m$, $\Sigma my = \bar{y}\Sigma m$. Similar equations must apply to the rates of change, and therefore to the accelerations, so that

$$\ddot{x} = \ddot{\bar{x}} + \ddot{x}', \qquad \Sigma m\ddot{x}' = 0,$$
$$\ddot{y} = \ddot{\bar{y}} + \ddot{y}', \qquad \Sigma m\ddot{y}' = 0.$$

Hence
$$\Sigma m(x\ddot{y} - y\ddot{x}) = \Sigma m(x'\ddot{y}' - y'\ddot{x}') + \Sigma m(\bar{x}\ddot{\bar{y}} - \bar{y}\ddot{\bar{x}})$$
$$+ \Sigma mx' \cdot \ddot{\bar{y}} - \Sigma my' \cdot \ddot{\bar{x}} + \Sigma m\ddot{y}' \cdot \bar{x} - \Sigma m\ddot{x}' \cdot \bar{y}.$$

Every term in the second line vanishes identically, and the first line becomes, in virtue of (3), (4) and (5),

$$\Sigma(xY - yX) = \Sigma m(x'\ddot{y}' - y'\ddot{x}') + \Sigma(\bar{x}Y - \bar{y}X),$$

and therefore

$$\Sigma m(x'\ddot{y}' - y'\ddot{x}') = \Sigma(x - \bar{x})Y - \Sigma(y - \bar{y})X$$
$$= \Sigma(x'Y - y'X) = \mu', \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

where μ' is the moment of the external forces about the centre of gravity of the body. This equation is of the same form as (3), but there is the important difference that, whereas in (3) the coordinates are referred to an origin which is fixed in space (or can be so taken for the purpose of the problem) but not in general fixed in the body, in (7) the coordinates are referred to the centre of gravity, which is a point fixed in the body and not in general fixed in space.

220. From this identity of form it can be inferred that the rotational motion of the body under given external forces can be found as though the centre of gravity were fixed, while the motion of the centre of gravity itself is independently determined by equations (4) and (5). In fact, just as (6) has been derived from (3), so from (7) may be derived

$$\ddot{\theta} \cdot \Sigma mr'^2 = \Sigma(x'Y - y'X) = \mu', \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

where $\ddot{\theta}$ is the angular acceleration, or rate of increase of the angular velocity, of the body about its centre of gravity, and r' is the perpendicular distance of the particle m from the axis drawn through the centre of gravity at right angles to the plane Oxy, to which the motion is parallel.

It is the two-dimensional character of the motion assumed which makes this axis virtually fixed in the body. Otherwise in completely general motion the body may be expected to heel over even in the absence of applied forces, and the instantaneous axis of rotation takes successively different positions both in the body and in space. Thus the more special motion considered here gives no idea of the difficulties which will be met with in the general problem of rigid dynamics. In the dynamics of a particle the transition from two to three dimensions is relatively simple. In the case of rigid bodies the increase in complexity is considerable because, apart from the motion of the centre of gravity, a knowledge of the orientation of the body depends upon the relation at any given instant between a set of three axes fixed in the body and a set of

three axes fixed in space. With the present limitations of the problem, two cases can be admitted:

(1) The body is only capable of rotating about a fixed axis, as in the case of a pendulum on a knife-edge. The equation (6) is then all that is needed for a solution, unless the pressure on the axis is also required.

(2) Though no point is fixed, the motion of every point of the body is parallel to a certain plane, the position of which is generally obvious from the conditions of the problem, as in the case of a sphere rolling down an inclined plane. The motion of the centre of gravity is then determined by (4) and (5), and the motion about an axis through this centre and perpendicular to the plane of motion is determined by (8).

221. Equations (4) and (5) apply to the mass M concentrated in the centre of gravity of the body as to a particle, and present nothing new. Equations (6) and (8) require a knowledge of Σmr^2 or $\Sigma mr'^2$, quantities which are called *moments of inertia* of the body about the corresponding axes, and which clearly depend on the shape of the body and the distribution of mass within its figure. The perpendicular distances r or r' from the axis acquire a definite meaning only when the elementary particles m are reduced indefinitely in size. Thus the summation for all the particles of the body involves in general the processes of integration, though there are some simple cases which can be treated by elementary methods.

The total mass of the body being M, Σmr^2 is of the form Mk^2, where k is a certain length. This length is called the *radius of gyration* of the body about the axis.

A plane sheet of matter, of small thickness, is called a *lamina*. When two rectangular axes OX, OY are taken in its plane, the corresponding moments of inertia are

$$I_x = \Sigma my^2, \qquad I_y = \Sigma mx^2;$$

and therefore

$$I_x + I_y = \Sigma m(x^2 + y^2) = \Sigma mr^2 = I_z,$$

where I_z is the moment of inertia about the axis OZ perpendicular to the plane of the lamina.

222. In the case of a thin circular ring of radius a every element of mass is situated at the same distance a from the central axis at

right angles to the plane of the ring. Hence the moment of inertia about this axis OZ is

$$I_z = \Sigma ma^2 = Ma^2.$$

If axes OX, OY are taken along two perpendicular diameters of the ring, symmetry shows that for the moments about these diameters

$$I_x = I_y = \tfrac{1}{2}(I_x + I_y) = \tfrac{1}{2} I_z = \tfrac{1}{2} Ma^2.$$

For a uniform thin circular disc of radius a, let ρ be the mass per unit area, so that $M = \pi\rho a^2$. The area between concentric circles of radius r and $r + \delta r$ is $2\pi r \,.\, \delta r$ and the corresponding mass $2\pi\rho r \,.\, \delta r$. Hence the moment of inertia about the central axis OZ at right angles to the disc is

$$I_z = \Sigma\, 2\pi\rho r \,.\, r^2 \delta r = 2\pi\rho \int_0^a r^3 dr$$

$$= 2\pi\rho \,.\, \left[\tfrac{1}{4} r^4\right]_0^a = \tfrac{1}{2}\pi\rho a^4 = \tfrac{1}{2} Ma^2.$$

Equally for the disc as for the ring, the moments about all diameters are equal by symmetry and are given by

$$I_x = I_y = \tfrac{1}{2}(I_x + I_y) = \tfrac{1}{2} I_z = \tfrac{1}{4} Ma^2.$$

These results can clearly be extended to a right circular cylinder to obtain the moments of inertia about the axis, for they apply in detail and in the sum to the thin sections into which the cylinder can be divided by planes parallel to the base. Thus $\frac{1}{2}Ma^2$ is the moment of inertia of a uniform solid cylinder of radius a and mass M, while Ma^2 is the moment of a thin cylindrical sheet with open ends.

223. The moment of inertia of a thin spherical shell is clearly the same about all diameters, in particular about three diameters mutually at right angles. Take these as axes OX, OY, OZ, so that x, y, z the coordinates of any point on the sphere are three sides of a rectangular block of which three edges lie along OX, OY, OZ

The diagonal of the block being a, the radius of the sphere,

$$x^2 + y^2 + z^2 = a^2.$$

The equal moments about the three axes can be expressed in the forms

$$I = \Sigma m(x^2 + y^2) = \Sigma m(y^2 + z^2) = \Sigma m(z^2 + x^2)$$

$$= \tfrac{1}{3}\Sigma m(2x^2 + 2y^2 + 2z^2)$$

$$= \tfrac{2}{3}\Sigma ma^2 = \tfrac{2}{3} Ma^2,$$

where $M = \Sigma m$ is the total mass of the shell.

To deduce the moment of inertia of a uniform solid sphere about a diameter, divide the sphere into thin concentric shells. The volume of the shell whose inner and outer radii are r and $r+\delta r$ is $4\pi r^2 . \delta r$, and its mass is $4\pi\rho r^2 . \delta r$, ρ being the density. Hence the moment of inertia of the solid sphere is

$$I = \Sigma \tfrac{2}{3} r^2 . 4\pi\rho r^2 . \delta r = \frac{8}{3}\pi\rho \int_0^a r^4 dr$$

$$= \frac{8}{3}\pi\rho \left[\tfrac{1}{5} r^5\right]_0^a = \frac{8}{15}\pi\rho a^5.$$

But the mass of the sphere is $M = \frac{4}{3}\pi\rho a^3$, and therefore

$$I = \tfrac{2}{5} M a^2$$

is the moment of inertia about any diameter.

224. The preceding results refer only to bodies of the simplest shape, and then only to particular axes. It is, however, possible, when the moment about one axis is known, to deduce the moment about any parallel axis by a theorem now to be proved.

Let I be the moment of inertia of the body about the axis OZ perpendicular to the plane of the rectangular axes OX, OY. Let G be the centre of gravity of the body, or its projection on the plane OXY. As in § 219 let $(\bar{x}, \bar{y})$ be the coordinates of G referred to O as origin, (x, y) the coordinates of a particle m (which need not be in the plane OXY) referred to the same axes, and (x', y') the coordinates of m referred to parallel axes through G. Thus

$$x = \bar{x} + x', \qquad \Sigma m x' = 0,$$

$$y = \bar{y} + y', \qquad \Sigma m y' = 0.$$

Hence it follows that

$$I = \Sigma m(x^2 + y^2)$$

$$= \Sigma m(x'^2 + y'^2) + \Sigma m(\bar{x}^2 + \bar{y}^2) + 2\Sigma m x' . \bar{x} + 2\Sigma m y' . \bar{y}$$

$$= I_0 + M . GO^2,$$

where $I_0 = \Sigma m(x'^2 + y'^2)$ is the moment of inertia about an axis through G parallel to OZ, and GO is the perpendicular distance of G from the axis OZ, about which the moment of inertia is I. This important result is perfectly general.

It is thus only necessary to know the moment of inertia of a body about all the lines passing through a particular point, the centre of gravity for example, to deduce the moment about any axis whatever. For all parallel axes the moment is clearly least about that one which passes through the centre of gravity. It follows as a corollary

that if two bodies or material systems are dynamically equivalent, that is, *equi-momental* with respect to all axes, two conditions must be fulfilled: (1) the two systems must have the same centre of gravity, (2) they must have the same mass M.

225. The moment of inertia of a thin uniform rod OA about an axis through the end O at right angles to its length is easily found. Let ρ be the mass per unit length, and a the length, so that $M=\rho a$ is the mass of the rod. The element of the rod between the distances x and $x+\delta x$ from O has the length δx and the mass $m=\rho\delta x$. Hence the moment of inertia required is

$$I=\Sigma\rho\delta x\,.\,x^2=\rho\int_0^a x^2\,dx$$

$$=\rho\left[\tfrac{1}{3}x^3\right]_0^a=\tfrac{1}{3}\rho a^3=\tfrac{1}{3}Ma^2.$$

Let I_0 be the moment of inertia about a parallel axis through the centre of gravity, which is the middle point of the rod Then

$$I=\tfrac{1}{3}Ma^2=I_0+M(\tfrac{1}{2}a)^2,$$

$$I_0=\tfrac{1}{3}Ma^2-\tfrac{1}{4}Ma^2=\tfrac{1}{12}Ma^2.$$

This result is otherwise evident, because the whole rod is divided at the centre of gravity into two equal rods, each of mass $\frac{1}{2}M$ and length $\frac{1}{2}a$.

Similarly the moment of inertia of a parallelogram $ABCD$ about a side $AB=a$ may be found. Draw $OX=\beta$ perpendicular to the

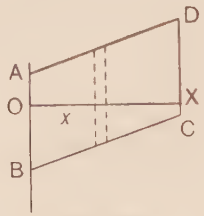

FIG. 129.

sides AB, CD, and divide the parallelogram into equal narrow strips, each of mass m, with sides parallel to AB. The matter in each strip is at the same distance x from AB, and the moment of inertia of the whole uniform sheet about AB is

$$I=\Sigma mx^2=\tfrac{1}{3}M\beta^2,$$

for the result is clearly the same as if OX were a uniformly loaded rod with the same total mass. If the side $BC=b$, and the angle

$ABC=\alpha$, $\beta=b\sin\alpha$, and the moment of the parallelogram may be written

$$I=\tfrac{1}{3}Mb^2\sin^2\alpha.$$

A thin rod may be treated as a limiting case, so that its moment of inertia about any line through one end, making an angle α with the length a of the rod, is $\tfrac{1}{3}Ma^2\sin^2\alpha$.

Lines parallel to the sides of a parallelogram through the centre of gravity divide the whole into four equal parallelograms, each of mass $\tfrac{1}{4}M$ and sides $\tfrac{1}{2}a$, $\tfrac{1}{2}b$. Hence the moment of inertia about an axis through the centre parallel to the side a is

$$I=4\,.\,\tfrac{1}{3}(\tfrac{1}{4}M)\,.\,(\tfrac{1}{2}b)^2\sin^2\alpha$$
$$=\tfrac{1}{12}Mb^2\sin^2\alpha.$$

In the case of a rectangle with sides a, b, the angle $\alpha=90°$. Hence the moment of inertia about the bisector of the figure parallel to the side a is $\tfrac{1}{12}Mb^2$, and that about the bisector parallel to the side b is $\tfrac{1}{12}Ma^2$. These axes being at right angles, it follows that the moment of inertia about the central axis normal to the plane of the figure is $\tfrac{1}{12}M(a^2+b^2)$.

The moment of inertia of a uniform rectangular block with sides a, b, c about the central axis parallel to the side c can be inferred from the last result. The solid can be divided into equal thin parallel sections, each of mass m, at right angles to the side c. The moment of each is $\tfrac{1}{12}m(a^2+b^2)$ about the central axis parallel to the side c. Hence for the whole solid about the same axis

$$I=\Sigma\,.\,\tfrac{1}{12}m(a^2+b^2)=\tfrac{1}{12}M(a^2+b^2).$$

226. From the result for a parallelogram the moment of inertia of a triangle ABC about any side BC can be deduced. Complete

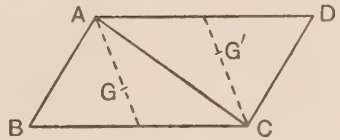

FIG. 130.

the parallelogram $ABCD$, so that ABC, CDA are equal similar triangles, each of mass M. Let G, G' be the centres of gravity of these triangles. If p is the perpendicular distance of A from BC, $\tfrac{1}{3}p$ is the perpendicular distance of G from BC and of G' from AD. Let I be the moment of inertia of each triangle about an axis through

its centre of gravity parallel to BC or AD. Then $I+\frac{1}{9}Mp^2$ is the moment of ABC about BC and $I+\frac{4}{9}Mp^2$ is the moment of ACD about BC, and the two together give the moment of the parallelogram. Hence

$$I+\tfrac{1}{9}Mp^2+I+\tfrac{4}{9}Mp^2=\tfrac{1}{3}.2Mp^2,$$
$$I=\tfrac{1}{2}Mp^2(\tfrac{2}{3}-\tfrac{5}{9})=\tfrac{1}{18}Mp^2.$$

Thus the moment of inertia of the triangle ABC about its side BC is

$$I_1=I+\tfrac{1}{9}Mp^2=\tfrac{1}{6}Mp^2.$$

227. The moment of inertia of the triangle about any line in its plane can be deduced. Let α, β, γ be the perpendiculars from

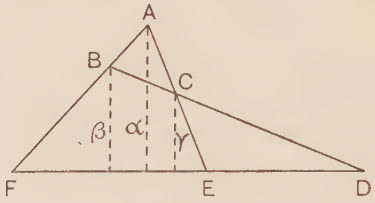

FIG. 131.

A, B, C on the line DEF, and let $I(ABC)$ represent the moment of inertia of ABC about this line. Then from the figure it is seen that

$$I(ABC)=I(AFE)-I(BDF)+I(CED),$$
$$=\tfrac{1}{6}(\alpha^2.AFE-\beta^2.BFD+\gamma^2.CED),$$

where the triangles stand for their masses, so that $ABC=M$. Hence

$$I(ABC)=\tfrac{1}{6}M\Big(\alpha^2.\frac{AFE}{ABC}-\beta^2.\frac{BFD}{ABC}+\gamma^2.\frac{CED}{ABC}\Big)$$
$$=\tfrac{1}{6}M\Big(\alpha^2.\frac{AF}{AB}.\frac{AE}{AC}-\beta^2.\frac{BF.BD}{BA.BC}+\gamma^2.\frac{CE}{CA}.\frac{CD}{CB}\Big)$$
$$=\tfrac{1}{6}M\Big(\alpha^2.\frac{\alpha}{\alpha-\beta}.\frac{\alpha}{\alpha-\gamma}-\beta^2.\frac{\beta}{\alpha-\beta}.\frac{\beta}{\beta-\gamma}+\gamma^2.\frac{\gamma}{\alpha-\gamma}.\frac{\gamma}{\beta-\gamma}\Big)$$
$$=\frac{M\{\alpha^4(\beta-\gamma)+\beta^4(\gamma-\alpha)+\gamma^4(\alpha-\beta)\}}{6(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}$$
$$=\tfrac{1}{6}M(\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta),$$

for the factors in the denominator are clearly factors of the numerator and the result is easily verified. But this result can also be written in the form

$$I(ABC)=\tfrac{1}{3}M[\{\tfrac{1}{2}(\beta+\gamma)\}^2+\{\tfrac{1}{2}(\gamma+\alpha)\}^2+\{\tfrac{1}{2}(\alpha+\beta)\}^2],$$

and $\frac{1}{2}(\beta+\gamma)$, $\frac{1}{2}(\gamma+\alpha)$, $\frac{1}{2}(\alpha-\beta)$ are the perpendicular distances of the middle points of the sides of the triangle ABC from the line DF.

Hence the moment of a triangle of mass M about any line in its plane is the same as that of three particles, each of mass $\frac{1}{3}M$, placed at the middle points of its sides, about the same line. The triangle and the three particles have the same total mass and the same centre of gravity, and are in fact equi-momental systems.

228. It has been seen that the moment of inertia of a body about any axis is readily deduced from that about a parallel axis passing through a particular point. The moments about axes through a given point are also related in a certain way, so that a complete knowledge of them depends only on a limited number of data. The general theory involves solid geometry and is beyond the present scope, but for a lamina, or thin sheet of matter in a plane, such a theory can be given.

Let OX, OY be two rectangular axes in the plane of the lamina. Let (x, y) be the coordinates of P, at which a particle of mass m is

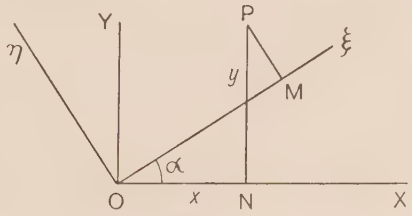

FIG. 132.

situated. Let $O\xi$ be an axis making the angle α with OX. If M, N are the feet of the perpendiculars from P on $O\xi$, OX, the projection of ON, NP on PM gives

$$PM = \eta = y\cos\alpha - x\sin\alpha.$$

Thus the moment of inertia of the body about $O\xi$ is

$$\begin{aligned} I_\xi &= \Sigma m\eta^2 = \Sigma m(y\cos\alpha - x\sin\alpha)^2 \\ &= \cos^2\alpha\,.\,\Sigma my^2 + \sin^2\alpha\,.\,\Sigma mx^2 - \sin 2\alpha\,\Sigma mxy, \end{aligned}$$

which involves not only $I_x = \Sigma my^2$ and $I_y = \Sigma mx^2$, but also

$$P_{xy} = \Sigma mxy.$$

When the latter quantity is known, I_ξ is also known for any given direction α of the axis $O\xi$.

The quantity $P_{xy} = \Sigma mxy$ is called the *product of inertia* of the body with respect to the rectangular axes OX, OY. For greater generality, the coordinates x, y can be defined as the perpendiculars on planes OYZ, OXZ at right angles respectively to OX, OY.

The projection of ON, NP on OM gives

$$OM=\xi=x\cos\alpha+y\sin\alpha.$$

Thus the product of inertia with respect to $O\xi$, $O\eta$ is

$$\begin{aligned}P_{\xi\eta}&=\Sigma m\xi\eta=\Sigma m(x\cos\alpha+y\sin\alpha)(y\cos\alpha-x\sin\alpha)\\&=(\cos^2\alpha-\sin^2\alpha)\Sigma mxy+\sin\alpha\cos\alpha(\Sigma my^2-\Sigma mx^2)\\&=P_{xy}\cos 2\alpha+\tfrac{1}{2}(I_x-I_y)\sin 2\alpha.\end{aligned}$$

Hence $P_{\xi\eta}=0$ when

$$\tan 2\alpha=\frac{2P_{xy}}{I_y-I_x}.$$

In general this gives two values α and $\alpha+90°$, and definite directions for the axes $O\xi$, $O\eta$, about which the product of inertia vanishes. The axes so determined are called the *principal axes* of the body at the point O.

The only exception arises when $P_{xy}=0$ and $I_x=I_y$. In this case $P_{\xi\eta}=0$ and $I_\xi=I_x=$ constant, for all directions of the axes.

229. When OX, OY are the principal axes, $P_{xy}=0$ and

$$I_\xi=I_x\cos^2\alpha+I_y\sin^2\alpha.$$

If a distance r is taken along $O\xi$, the coordinates of the point

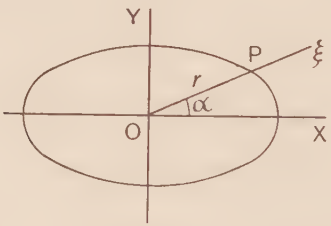

FIG. 133.

reached are $x=r\cos\alpha$, $y=r\sin\alpha$. Thus the ellipse

$$I_x.x^2+I_y.y^2=1,$$

which is called the *momental ellipse*, has a central radius r in the direction $O\xi$ given by

$$I_x\cos^2\alpha+I_y\sin^2\alpha=1/r^2.$$

Hence I_ξ is represented by $1/r^2=1/OP^2$, and in this way the momental ellipse exhibits the variation of the moment of inertia as the axis turns in the plane.

Referred to any axes, the momental ellipse has the equation

$$I_x.x^2+I_y.y^2-2P_{xy}.xy=1=I_\xi.r^2,$$

and the problem of finding the principal axes, about which the moments are respectively the greatest and the least, is the same as that of finding the axes of this ellipse.

In the exceptional case, noticed above, the momental ellipse is a circle, and every axis is a principal axis.

230. The axis OX will not in general be a principal axis at O, but it may be such an axis at a point H at a distance h from O.

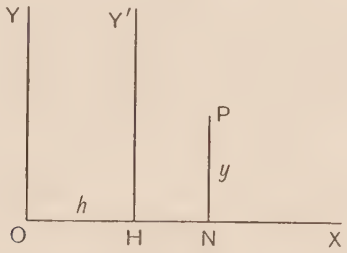

Fig. 134.

This is the case if the product of inertia with respect to the rectangular axes HX, HY' vanishes, or

$$\Sigma my(x-h)=0.$$

This requires that

$$h=\Sigma mxy/\Sigma my=P_{xy}/M\bar{y},$$

where P_{xy} is the product of inertia with respect to OX, OY and $\bar{y}$ is the ordinate of the centre of gravity G. Thus H is a definite point on OX unless $\bar{y}=0$ and G is situated on OX. In that case

$$\Sigma my(x-h)=\Sigma mxy,$$

and is constant for all values of h. Hence a line which passes through the centre of gravity is a principal axis at all points or at none; if it does not pass through the centre of gravity it is a principal axis at one definite point.

231. It remains to find the moment of inertia of a lamina about an axis which does not lie in the plane. Let OX be the projection of the axis OQ on the plane of the lamina, and let OY be the perpendicular axis in this plane. From P, the position of a small mass m, draw PM perpendicular to OX, and draw MN perpendicular to OQ. Now OY is perpendicular to the plane QOX, and therefore PM, parallel to OY, is perpendicular to MN in the plane QOX. Also OQ, being perpendicular to MN and to OY or PM, is perpendicular to the plane PMN and therefore to PN. Hence the moment of

inertia of the lamina in the plane XOY about OQ, making with OX the angle α, is

$$\begin{aligned} I = \Sigma m \,.\, PN^2 &= \Sigma m(PM^2 + MN^2) \\ &= \Sigma m(y^2 + x^2 \sin^2 \alpha) \\ &= I_x + I_y \sin^2 \alpha. \end{aligned}$$

Thus in the case of a lamina, the moment about any axis can be derived from a knowledge of the moments about axes in the plane

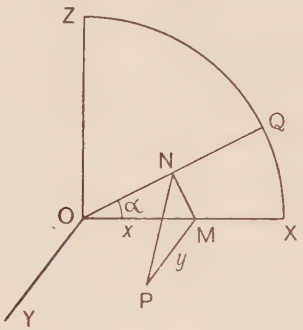

FIG. 135.

of the lamina. If $\alpha = 0$, $I = I_x$ naturally, and if $\alpha = 90°$, $I = I_x + I_y$, as already remarked.

As a corollary, if two systems in a plane are equi-momental for all axes in that plane, they are equi-momental for all axes without restriction. This completes the proof that a triangle and the system of three particles, which have been found to be dynamically equivalent, are equi-momental in the fullest sense.

232. The calculation of the moment of inertia of a solid body about any axis is not in general possible by elementary methods. Exceptionally easy cases have already been illustrated by the sphere and the right circular cylinder about its axis of figure. A few more examples will be given, and first the moment of a uniform cylinder about a diameter of the base.

Let the mass of the cylinder be M, the density ρ, the radius a and the height H, so that $M = \pi\rho a^2 H$. The cylinder being divided into thin sections parallel to the base, consider the section of which the ends are at heights h and $h + \delta h$ above the base. The mass of this section is $\pi\rho a^2 . \delta h$, its moment about its own diameter is $\frac{1}{4}\pi\rho a^4 . \delta h$ and its moment about a parallel diameter of the base is

$\frac{1}{4}\pi\rho a^4.\,\delta h+\pi\rho a^2h^2.\,\delta h$. Hence the required moment of the whole cylinder is

$$\begin{aligned} I&=\Sigma\{\tfrac{1}{4}\pi\rho a^4.\,\delta h+\pi\rho a^2h^2.\,\delta h\}\\ &=\tfrac{1}{4}\pi\rho a^4H+\pi\rho a^2\int_0^H h^2dh\\ &=\tfrac{1}{4}Ma^2+\pi\rho a^2\Big[\tfrac{1}{3}h^3\Big]_0^H\\ &=M(\tfrac{1}{4}a^2+\tfrac{1}{3}H^2), \end{aligned}$$

while the moment of inertia about the axis of the cylinder has been found to be $\frac{1}{2}Ma^2$.

233. The moment of inertia of a right circular cone may be calculated, first about its axis and secondly about a diameter of its base.

Let the mass of the cone be M, the density ρ, the radius of the base a and the height H, so that $M=\frac{1}{3}\pi\rho a^2H$. If a plane be drawn through the cone parallel to the base at a depth h below the vertex, the section will be a circle of radius ah/H. Hence the mass between two planes at depths h and $h+\delta h$ below the vertex will be

$$\pi\rho a^2h^2.\,\delta h/H^2,$$

and the moment of this thin section about the axis of the cone will be $\frac{1}{2}\pi\rho a^4h^4.\,\delta h/H^4$. Hence the moment of the whole cone about its axis is

$$\begin{aligned} I_1&=\Sigma\,.\,\tfrac{1}{2}\pi\rho a^4h^4.\,\delta h/H^4\\ &=\tfrac{1}{2}\pi\rho a^4H^{-4}\int_0^H h^4dh\\ &=\tfrac{3}{2}Ma^2H^{-5}\Big[\tfrac{1}{5}h^5\Big]_0^H=\tfrac{3}{10}Ma^2. \end{aligned}$$

When the moment of the cone about a diameter of its base is required, the moment of the thin circular section considered above about its own diameter is $\frac{1}{4}\pi\rho a^4h^4.\,\delta h/H^4$. The distance of its centre of gravity from the parallel diameter of the base of the cone is $H-h$. Hence the required moment of inertia of the whole cone about this diameter is

$$\begin{aligned} I_2&=\Sigma\tfrac{1}{4}\pi\rho a^4h^4.\,\delta h/H^4+\Sigma\pi\rho a^2h^2(H-h)^2.\,\delta h/H^2\\ &=\tfrac{1}{4}\pi\rho a^4H^{-4}\int_0^H h^4dh+\pi\rho a^2H^{-2}\Big\{H^2\int_0^H h^2dh-2H\int_0^H h^3dh+\int_0^H h^4dh\Big\}\\ &=\tfrac{1}{20}\pi\rho a^4H+\pi\rho a^2H^{-2}\{\tfrac{1}{3}H^5-\tfrac{1}{2}H^5+\tfrac{1}{5}H^5\}\\ &=\tfrac{1}{20}\pi\rho a^4H+\tfrac{1}{30}\pi\rho a^2H^3=\tfrac{3}{20}Ma^2+\tfrac{1}{10}MH^2\\ &=\tfrac{1}{20}M(3a^2+2H^2). \end{aligned}$$

About a parallel axis through the centre of gravity of the cone, at a height $\frac{1}{4}H$ above the base, the moment of inertia is

$$I' = I_2 - \tfrac{1}{16}MH^2 = \tfrac{3}{80}M(4a^2 + H^2).$$

In a dynamical problem it is usually convenient to leave a moment of inertia expressed in the general form Mk^2 till the latest possible stage in the development.

234. The simple pendulum is clearly a fiction which can only be realized very imperfectly in practice. An ordinary pendulum is a rigid body supported on a horizontal knife-edge, about which it can rotate freely. It is sometimes called a compound pendulum, and it illustrates in the simplest way the motion of a rigid body about a fixed axis.

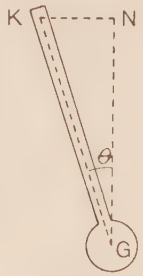

FIG. 136.

Let the mass of the pendulum be M, the centre of gravity G, the knife-edge on the pendulum K and the distance KG equal to h. Let Mk^2 be the moment of inertia of the pendulum about a horizontal axis through G parallel to the knife-edge, so that $M(h^2+k^2)$ is the moment of inertia about the knife-edge itself. When θ is the inclination of KG to the vertical GN, the moment of the weight Mg about K is $Mg \,.\, KN = Mgh \sin\theta$. Hence the angular motion of the pendulum about K is given by the equation

$$M(k^2+h^2)\ddot{\theta} = -Mgh \sin\theta,$$

the sense of the moment being opposite to that in which θ increases.

Thus

$$\ddot{\theta} + \frac{gh}{k^2+h^2}\sin\theta = 0.$$

The corresponding equation for a simple pendulum of length L, obtained by putting $h=L$, $k=0$, is

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0.$$

Hence the motion of the rigid pendulum is the same as that of a simple pendulum of length L, if

$$L=(h^2+k^2)/h,$$

and this is called the length of the *simple equivalent pendulum.*

When the amplitude α of a swing is very small it is possible to substitute the approximate equation, giving simple harmonic motion,

$$\ddot{\theta}+\frac{gh}{h^2+k^2}\theta=0,$$

for which the solution is

$$\theta=\alpha\sin\left\{\left(\frac{gh}{h^2+k^2}\right)^{\frac{1}{2}}(t-t_0)\right\},$$

where α is an arbitrary small angle, and t_0 is the time of passing the lowest position. The limiting period of a complete swing is

$$T=2\pi\left(\frac{L}{g}\right)^{\frac{1}{2}}=2\pi\left(\frac{h^2+k^2}{gh}\right)^{\frac{1}{2}}.$$

235. The relation between the rigid pendulum and the simple equivalent pendulum can be represented graphically.

The position of the knife-edge K is called the *centre of suspension.*

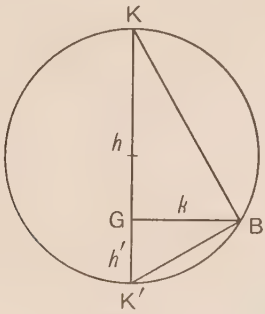

FIG. 137.

Through the centre of gravity G draw $GB=k$ at right angles to $GK=h$, so that GB is the radius of gyration of the pendulum about the horizontal axis through G. Draw BK' at right angles to BK, to meet KG in K'. Then $k^2=hh'$, if $GK'=h'$. Hence

$$KK'=h+h'=\frac{h^2+k^2}{h}=L,$$

so that KK' is the length of the simple equivalent pendulum. This point K' on the axis of the pendulum KG is called the *centre of oscillation.*

But it is evident that if the pendulum were inverted, and the knife-edge were placed at K', the same construction would lead to the point K as the centre of oscillation. Thus the points K, K' have a reciprocal relation as centres of suspension and oscillation. The length of the simple equivalent pendulum is $KK'=L$ in either case, and the period of oscillation is the same whether K or K' is made the centre of suspension. It is also clear that the circle on KK' as diameter passes through B, the point at a distance from the axis KG equal to the radius of gyration.

236. For the accurate determination of g by observing T, the equation

$$T=2\pi\left(\frac{h^2+k^2}{gh}\right)^{\frac{1}{2}}, \qquad g=4\pi^2.\frac{h^2+k^2}{hT^2},$$

even when corrected for the effect of finite amplitude (cf. § 134), is by itself scarcely suitable. The centre of gravity G is not a marked point enabling the distance h to be measured accurately. Even if the pendulum has a simple geometrical shape, the calculation of the moment of inertia Mk^2 is liable to an unknown error owing to the want of uniformity in the material. On the contrary, the equation suggests a suitable method for determining the moment of inertia of a body, when g is known, by mounting it as a pendulum.

If, however, the pendulum is provided with two parallel knife-edges at distances h, h' on opposite sides of the centre of gravity, and T, T' are the observed periods of swing about these supports,

$$T^2=\frac{4\pi^2}{gh}(h^2+k^2), \quad T'^2=\frac{4\pi^2}{gh'}(h'^2+k^2).$$

Hence $$T^2-T'^2=\frac{4\pi^2}{ghh'}(h'-h)(k^2-hh'),$$

$$T^2+T'^2=\frac{4\pi^2}{ghh'}(h'+h)(k^2+hh'),$$

and $$\frac{T^2+T'^2}{h+h'}+\frac{T^2-T'^2}{h-h'}=\frac{8\pi^2}{g}.$$

The unknown moment of inertia has now been eliminated, and g can be determined by observing T, T' and measuring h and h'.

The first term on the left can always be determined with high precision, because it involves the sum of h and h', and this sum is simply the distance between the knife-edges without reference to the position of the centre of gravity.

The second term on the left can only be determined less precisely because the difference $h-h'$ does involve the position of the

unmarked point G. It is therefore necessary to make this term vanish, or relatively small in comparison with the first. Now $T^2-T'^2=0$ in two circumstances: (1) when $h=h'$, (2) when $hh'=k^2$.

The first condition, $h=h'$, will arise when the pendulum has symmetry of mass on either side of the centre of gravity. This condition must be avoided, for then $(T^2-T'^2)/(h-h')$ tends to a finite limit which involves the unknown moment of inertia.

In the second condition the knife-edges occupy alternately the corresponding positions of centre of suspension and centre of oscillation. When by adjustment of the knife-edges the periods T and T' tend to equality, h and h' tend to a finite difference.

This explains the essential features of Kater's pendulum, which is provided with one fixed knife-edge and also with a second knife-edge which can be adjusted in position until the period of oscillation about the two is identical or very nearly so. An exact identity need not be established, for the pendulum has no symmetry of mass. The second term then gives a small correction to the first, and the divisor $h-h'$, without being determined with the same precision as $h+h'$, can be found accurately enough for this purpose. Hence the absolute value of g can be determined.

237. In the case of the simple pendulum the value of the principle of energy has been seen, as a means of obtaining the velocity without any approximation. It is now necessary to calculate the kinetic energy of a rigid body in two-dimensional motion.

Let (x, y) be the coordinates of a small mass m referred to any fixed axes. Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of gravity G, the whole mass of the body being M. If (x', y') are the coordinates of m relative to parallel axes through G,

$$x=\bar{x}+x', \quad y=\bar{y}+y', \quad \Sigma mx'=\Sigma my'=0,$$

as before, and

$$\dot{x}=\dot{\bar{x}}+\dot{x}', \quad \dot{y}=\dot{y}+\dot{y}', \quad \Sigma m\dot{x}'=\Sigma m\dot{y}'=0,$$

are the corresponding relations for the velocities. Hence the kinetic energy of the body is

$$\begin{aligned}\Sigma\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2)&=\Sigma\tfrac{1}{2}m\dot{x}^2+\Sigma m\dot{x}'.\dot{\bar{x}}+\Sigma\tfrac{1}{2}m\dot{x}'^2\\&\qquad+\Sigma\tfrac{1}{2}m\dot{\bar{y}}^2+\Sigma m\dot{y}'.\dot{\bar{y}}+\Sigma\tfrac{1}{2}m\dot{y}'^2\\&=\Sigma\tfrac{1}{2}m(\dot{x}'^2+\dot{y}'^2)+\Sigma\tfrac{1}{2}m(\dot{\bar{x}}^2+\dot{\bar{y}}^2),\end{aligned}$$

the mixed product terms disappearing in the sum, as they will always do in the expansion of a quadratic expression of this kind.

The kinetic energy is thus divided into two parts, of which the one, $\Sigma\frac{1}{2}m(\dot{x}'^2+\dot{y}'^2)$, is the kinetic energy of the body relative to G, as though this point were at rest, while the other, $\frac{1}{2}M(\bar{x}^2+\bar{y}^2)$ is the kinetic energy of the whole mass M concentrated as a single particle at the centre of gravity. This division of the kinetic energy, though proved here only for two-dimensional motion, is in fact quite general.

If the angular velocity of the body is $\dot{\theta}$, the particle m moves in a circle of radius r' about G with the linear velocity $r'\dot{\theta}$. Hence if v is the linear velocity of G, the kinetic energy of the body becomes

$$T=\Sigma\tfrac{1}{2}mr'^2.\,\dot{\theta}^2+\tfrac{1}{2}Mv^2$$
$$=\tfrac{1}{2}Mk^2\dot{\theta}^2+\tfrac{1}{2}Mv^2,$$

where the first term, containing as a factor the moment of inertia about the axis through G, may be called the energy of rotation, and the second term may be called the energy of translation.

When the body moves about a fixed axis, the point G describes a circle about this axis, and its velocity is $v=h\dot{\theta}$, where h is the distance of G from the axis. Hence the kinetic energy of the body becomes

$$T=\tfrac{1}{2}M(h^2+k^2)\dot{\theta}^2.$$

The factor $M(h^2+k^2)$ is the moment of the body about the fixed axis.

238. In the case of the rigid pendulum swinging about the point of suspension K, $KG=h$ and the vertical depth of G below K is $h\cos\theta$ when KG makes the angle θ with the vertical. In this position the potential energy of the pendulum is $-Mgh\cos\theta$. Hence the equation of energy is

$$\tfrac{1}{2}M(h^2+k^2)\dot{\theta}^2-Mgh\cos\theta=\text{const.}$$

If α is the amplitude, this is the value of θ when $\dot{\theta}=0$. Hence the same constant is given by

$$0-Mgh\cos\alpha=\text{const.},$$

and therefore $$\tfrac{1}{2}M(h^2+k^2)\dot{\theta}^2=Mgh(\cos\theta-\cos\alpha),$$

or $$\dot{\theta}^2=\frac{2gh}{h^2+k^2}(\cos\theta-\cos\alpha).$$

This equation, like the corresponding equation for the simple pendulum, is exact.

It is now possible to complete the discussion of the rigid pendulum by finding the reaction at the knife-edge. Let H be the component of the thrust on the knife-edge in the direction GK, and E the

component at right angles to GK. The corresponding components of the weight Mg are $-Mg\cos\theta$ and $-Mg\sin\theta$. Thus

$$H - Mg\cos\theta, \quad E - Mg\sin\theta$$

are the components of all the external forces acting on the pendulum.

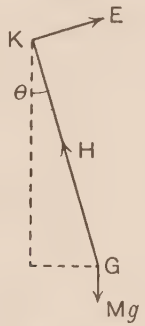

FIG. 138.

The centre of gravity G describes a circle about the fixed point K, and in this motion its acceleration has the components $h\dot{\theta}^2$ in the direction GK and $h\ddot{\theta}$ in the perpendicular direction. Hence equations (4) and (5) (p. 232) for the motion of G take the form

$$Mh\dot{\theta}^2 = H - Mg\cos\theta,$$
$$Mh\ddot{\theta} = E - Mg\sin\theta.$$

But $\dot{\theta}^2$ has been found above, and (§ 234)

$$\ddot{\theta} + \frac{gh}{h^2+k^2}\sin\theta = 0.$$

Hence
$$H = M(h\dot{\theta}^2 + g\cos\theta)$$
$$= Mg\left\{\frac{2h^2}{h^2+k^2}(\cos\theta - \cos\alpha) + \cos\theta\right\}$$
$$= \frac{Mg}{h^2+k^2}\{(3h^2+k^2)\cos\theta - 2h^2\cos\alpha\},$$

and
$$E = M(h\ddot{\theta} + g\sin\theta)$$
$$= \frac{Mg}{h^2+k^2}\cdot k^2\sin\theta.$$

These are the required components of the reaction at the point of support.

239. The motion of a body free to move in two dimensions is illustrated in the following example.

Ex. A lamina rests on a smooth horizontal plane, and a constant force F in a fixed direction in the plane is applied at a given point P. To find the motion.

Take the axis OX in the fixed direction of F. Let (x, y) be the coordinates of the centre of gravity G of the lamina, and let the line $GP=h$ make an angle θ with OX at the time t. Then, if M is the mass and k is the radius of gyration of the lamina about G, the equations of motion (4), (5) and (8) give

$$M\ddot{x}=F,$$
$$M\ddot{y}=0,$$
$$Mk^2\,.\,\ddot{\theta}=-hF\sin\theta.$$

Since F is constant, the first two equations define a constant acceleration, and if G starts from rest at O when $t=0$,

$$\bar{x}=\tfrac{1}{2}Ft^2/M, \quad \bar{y}=0.$$

The third equation, which defines the motion of the lamina about G, is of the same form as the equation of motion of a pendulum under gravity. Hence the centre of gravity of the lamina moves on a straight line parallel to F with constant acceleration, and the lamina oscillates about G with respect to this line like a pendulum. If initially the angle θ is small, the period of the oscillation approaches the value

$$T=2\pi k(M/hF)^{\frac{1}{2}}.$$

240. The same principles are illustrated by the motion of a uniform sphere rolling down a fixed rough plane.

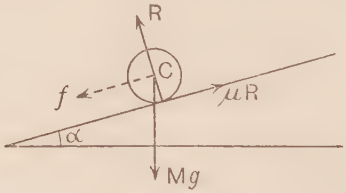

FIG. 139.

Let C be the centre, a the radius, M the mass of the sphere, and k its radius of gyration about C. Let α be the inclination of the plane, and μ the effective coefficient of friction. The tangential force is thus μR up the plane, when the normal reaction is R. The motion of the centre of gravity C is given by

$$0=R-Mg\cos\alpha,$$
$$Mf=Mg\sin\alpha-\mu R,$$

where f is the acceleration of C down the plane. Also the moment of the forces about C gives

$$Mk^2\ddot{\theta}=a\mu R,$$

where θ is the angle through which the sphere has turned from a definite position.

There are now two possibilities. In the first, the sphere rolls without slipping. Then $x=a\theta$ is the equal displacement of the centre of the sphere and of the point of contact on the plane, so that

$$f=\ddot{x}=a\ddot{\theta}.$$

From the above equations

$$Maf=Mag\sin\alpha-Mk^2\ddot{\theta},$$
$$(a^2+k^2)f=a^2g\sin\alpha,$$

and
$$f=a\ddot{\theta}=\frac{a^2}{a^2+k^2}\cdot g\sin\alpha.$$

Further,
$$R=Mg\cos\alpha,$$
$$\mu=\frac{k^2\ddot{\theta}}{ag\cos\alpha}=\frac{k^2}{a^2+k^2}\tan\alpha.$$

By giving the appropriate value to k the solution can be applied to a solid or hollow sphere, or to a cylinder rolling with its axis horizontal. In the actual problem of a uniform solid sphere,

$$k^2=0{\cdot}4a^2,$$
$$f=a\ddot{\theta}=\tfrac{5}{7}g\sin\alpha,$$

and
$$\mu=\tfrac{2}{7}\tan\alpha.$$

Now the coefficient of friction may be greater or less than this value. If it be greater this amount only comes into play and secures the possibility of pure rolling.

241. The second possibility, that slipping accompanies rolling, always arises when the coefficient of friction falls short of this necessary value. The equations of motion are as before, and may now be written :

$$R=Mg\cos\alpha,$$
$$f=g(\sin\alpha-\mu\cos\alpha),$$
$$\ddot{\theta}=\mu ag\cos\alpha/k^2.$$

In the previous case f and $\ddot{\theta}$ were related by the geometrical condition of pure rolling, and the necessary value of μ was deduced accordingly. Here, on the contrary, μ has its limiting value, and the linear and angular accelerations, f and $\ddot{\theta}$, are determined independently by these equations.

It has been assumed that the motion starts from rest, in which case pure rolling sets in if friction makes it possible. But the initial

conditions may be such that the body begins to roll only after the lapse of a certain time.

To simplify the problem let the plane be horizontal, so that $\alpha=0$ and

$$f=-\mu g, \quad \ddot{\theta}=\mu a g/k^2.$$

While these equations rule, if u, n are the initial velocities of translation and rotation measured in the corresponding sense, the velocities after a time t become, since f and $\ddot{\theta}$ are constant.

$$v=u-\mu g t,$$
$$\dot{\theta}=n+\mu a g t/k^2.$$

The tangential velocity at the point of contact is therefore

$$v-a\dot{\theta}=u-an-\mu g(1+a^2/k^2)t,$$

which vanishes in the case of pure rolling. Since friction always tends to reduce this velocity, both μ and $(u-an)$ must receive the same sign, and rolling sets in when

$$t=\frac{u-an}{\mu g}\cdot\frac{k^2}{a^2+k^2}.$$

Here μ is the limiting coefficient of friction. At this instant

$$v=u-\mu g t=\frac{a}{a^2+k^2}(au+k^2n),$$
$$\dot{\theta}=n+\frac{a}{a^2+k^2}(u-an)=\frac{1}{a^2+k^2}(au+k^2n).$$

The condition of pure rolling, $v=a\dot{\theta}$, is now established and subsequently maintained, the friction on a level plane ($\alpha=0$) tending effectively to zero.

Both μ and $(u-an)$ being of the same sign, which may be taken to be positive, $u>an$. Since $v=u-\mu g t$ while full friction acts, v cannot vanish and change sign unless u is positive. Suppose this to be the case. Now the final velocity v when rolling sets in is negative if $(au+k^2n)$ is negative. Hence when u is positive, n negative and

$$a^2n<au<-k^2n,$$

the sphere, being projected in one direction, ends by rolling in the opposite direction. This can be illustrated on the billiard table. In all other circumstances the stage of pure rolling is preceded by an adjustment between the velocity of projection and the velocity of rotation, which leaves the original direction of motion unchanged, provided the conditions of two-dimensional motion are observed throughout.

242. In the absence of sliding friction, or some similar agency taking energy out of its purely dynamical form, the equation of energy can be written down. If there is only one degree of freedom, this one equation is all that is required to solve the problem.

If, further, the motion is one of oscillation about a position of equilibrium the equation is comparable with that of a simple pendulum of length L, namely,

$$\tfrac{1}{2}ML^2\dot{\theta}^2 = MgL\,(\cos\theta - \cos\alpha),$$

or

$$\dot{\theta}^2 = 2g\,(\cos\theta - \cos\alpha)/L.$$

In *small* oscillations it is legitimate to make approximations. Thus, in this case of the simple pendulum,

$$\dot{\theta}^2 = g(a^2 - \theta^2)/L,$$

corresponding with

$$\ddot{\theta} = -g\theta/L,$$

whence the limiting period $T = 2\pi\sqrt{(L/g)}$.

243. *Ex.* To discuss the small oscillations of a sphere rocking in a vertical plane at the bottom of a spherical bowl.

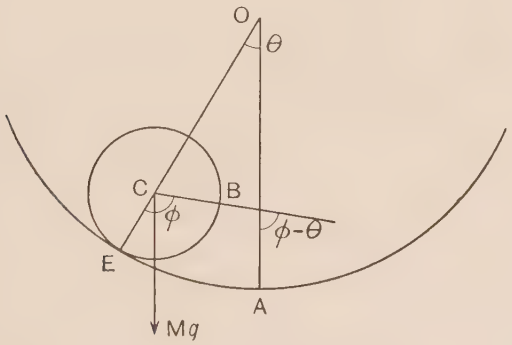

FIG. 140.

Let M be the mass and a the radius of the sphere, and b the radius of the bowl. In equilibrium the sphere rests with the point B in contact with A, the lowest point of the bowl. If E is the point of contact after rolling, the arcs EB, EA are equal, and if θ, ϕ are the angles EOA, ECB, $b\theta = a\phi$, or

$$\frac{\theta}{a} = \frac{\phi}{b} = \frac{\phi - \theta}{b - a}.$$

The rates of change of these angles are thus connected by the relations

$$\frac{\dot{\theta}}{a} = \frac{\dot{\phi}}{b} = \frac{\dot{\phi} - \dot{\theta}}{b - a},$$

and $\dot{\phi} - \dot{\theta}$ is the angular velocity of the sphere, being measured by the rotation of the radius CB from the vertical.

If k is the radius of gyration of the sphere about its centre C, which is also its centre of gravity, the kinetic energy of the sphere due to rotation is $\frac{1}{2}Mk^2(\dot{\phi}-\dot{\theta})^2$, and that due to the translation of C is

$$\tfrac{1}{2}M(b-a)^2\dot{\theta}^2,$$

for $(b-a)\dot{\theta}$ is the velocity of this point.

On the whole, the kinetic energy of the sphere is therefore

$$T=\tfrac{1}{2}Mk^2(\dot{\phi}-\dot{\theta})^2+\tfrac{1}{2}M(b-a)^2\dot{\theta}^2$$
$$=\tfrac{1}{2}M(b-a)^2\dot{\theta}^2(1+k^2/a^2).$$

The depth of C below O is $(b-a)\cos\theta$, and the potential energy of the sphere is $-Mg(b-a)\cos\theta$. Hence in the rolling motion

$$\tfrac{1}{2}M(b-a)^2(1+k^2/a^2)\dot{\theta}^2-Mg(b-a)\cos\theta=\text{const.},$$

and if $\theta=\alpha$ when the sphere comes to rest, or $\dot{\theta}=0$,

$$(b-a)(1+k^2/a^2)\dot{\theta}^2=2g(\cos\theta-\cos\alpha).$$

But the corresponding equation for the simple pendulum is

$$L\dot{\theta}^2=2g(\cos\theta-\cos\alpha).$$

Hence the sphere performs oscillations like a simple pendulum, and the length of the simple equivalent pendulum which gives the same period is

$$L=(b-a)(1+k^2/a^2).$$

244. When impulses are applied to a rigid body the state of motion is suddenly changed. The circumstances of this change are determined by a set of equations analogous to those which govern the motion of the body under ordinary forces, with the difference that they refer to an instantaneous effect and not to a process which needs to be traced in time.

A single small particle, of mass m, changes its velocity represented by the components (u, v) to one represented by (u', v'). Let (P_x, P_y) be the components of the impulse directly applied to it. In addition, as an element of the rigid body, it is subject to an impulsive reaction, with components (R_x, R_y), due to its connexion with the body. The change in its momentum is determined by the equations

$$m(u'-u)=P_x+R_x,$$
$$m(v'-v)=P_y+R_y.$$

When the equations of this type are summed for all the particles of the body, of total mass $M=\Sigma m$,

$$\Sigma m(u'-u)=M(\bar{u}'-\bar{u})=\Sigma P_x, \quad \ldots\ldots\ldots\ldots(9)$$
$$\Sigma m(v'-v)=M(\bar{v}'-\bar{v})=\Sigma P_y, \quad \ldots\ldots\ldots\ldots(10)$$

for the components of the reactions disappear in the sum, so that $\Sigma R_x = \Sigma R_y = 0$, and the total momentum of the body can be expressed in terms of $(\bar{u}, v)$, $(\bar{u}', v')$, the components of the velocity of the centre of gravity before and after the impulses are applied.

Also, if (x, y) are the coordinates of m at the moment when the impulse is received, the equations of the type of the first pair can be combined to give

$$\Sigma m\{x(v'-v)-y(u'-u)\} = \Sigma(xP_y - yP_x) + \Sigma(xR_y - yR_x),$$

or, since the equal opposed elements of (R_x, R_y) occur at the same point,

$$\Sigma m(xv'-yu') - \Sigma m(xv-yu) = \Sigma(xP_y - yP_x).$$

This last equation needs reducing to the appropriate form, as was done in the case of the corresponding equation for forces.

245. First, when the motion of the body is constrained by a fixed axis at right angles to the plane Oxy, passing through the fixed point O, every point describes a circle about this axis. The angular velocity changes from n to n'. If r is the perpendicular distance of the particle m from the axis, the linear velocity of m changes from rn with components $(-yn, xn)$ to rn' with components $(-yn', xn')$. Since these components are identical with (u, v), (u', v'), the last equation becomes

$$\Sigma m(x^2+y^2) \,.\, n' - \Sigma m(x^2+y^2) \,.\, n = \Sigma(xP_y - yP_x).$$

But $\Sigma m(x^2+y^2) = \Sigma mr^2 = Mk^2$, the moment of inertia of the body about the fixed axis. Hence

$$Mk^2(n'-n) = \Sigma(xP_y - yP_x), \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

where the right-hand side represents the moment of the impressed impulses about the axis. This is the required equation.

Secondly, when there is no fixed axis but the conditions of two-dimentional motion are satisfied, a transformation must be made to the centre of gravity of the body. For this the formulae are

$$\begin{aligned} x &= \bar{x} + x_1, & y &= \bar{y} + y_1, & \Sigma m x_1 &= \Sigma m y_1 = 0, \\ u &= \bar{u} + u_1 & v &= \bar{v} + v_1, & \Sigma m u_1 &= \Sigma m v_1 = 0, \\ u' &= \bar{u}' + u_1', & v' &= \bar{v}' + v_1', & \Sigma m u_1' &= \Sigma m v_1' = 0. \end{aligned}$$

Hence, on substitution,

$$\Sigma m(xv'-yu') = \Sigma m(x_1v_1'-y_1u_1') + \Sigma m(\bar{x}\,\bar{v}'-\bar{y}\,\bar{u}'),$$

when the terms such as $\bar{x}\Sigma mv_1'$ are omitted; similarly

$$\Sigma m(xv-yu) = \Sigma m(x_1v_1-y_1u_1) + \Sigma m(\bar{x}\,\bar{v}-\bar{y}\,\bar{u}).$$

The last equation of the preceding section now becomes

$$\Sigma m(x_1v_1' - y_1u_1') - \Sigma m(x_1v_1 - y_1u_1) + x\Sigma m(\bar{v}' - v) - y\Sigma m(\bar{u}' - u)$$
$$= \Sigma\{(\bar{x} + x_1)P_y - (\bar{y} + y_1)P_x\}.$$

But the equations

$$\Sigma m\,.\,(\bar{u}' - \bar{u}) = \Sigma P_x, \quad \Sigma m\,.\,(\bar{v}' - \bar{v}) = \Sigma P_y$$

of that section are true generally. Hence, more simply,

$$\Sigma m(x_1v_1' - y_1u_1') - \Sigma m(x_1v_1 - y_1u_1) = \Sigma(x_1P_y - y_1P_x).$$

Now this equation in terms of coordinates and components of velocity relative to the centre of gravity is of the same form as that relative to a fixed point or axis. Hence the same reduction can be applied and the result will clearly be

$$Mk_1^2(n' - n) = \Sigma(x_1P_y - y_1P_x), \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

where k_1 is the radius of gyration of the body about the axis through the centre of gravity.

246. The rigid pendulum, swinging in a vertical plane (which is also its plane of symmetry) about a fixed horizontal axis, illustrates the first case. Let K be the point of suspension, G the centre of gravity. Let $GK = h$, M be the mass of the pendulum and Mk^2 its moment of inertia about the horizontal axis through G. At the moment when the inclination of KG to the vertical is θ and the angular velocity is n, the pendulum receives an impulse P at the point H in KG, the direction of the blow making an angle α with the horizon. The effect is to change the angular velocity from n to n', and to produce an impulsive reaction R on the axis through K making an angle β with the horizon. The horizontal and vertical components of the velocity of G are $(hn\cos\theta, hn\sin\theta)$ before, and $(hn'\cos\theta, hn'\sin\theta)$ after the blow. Hence equations (9) and (10) give

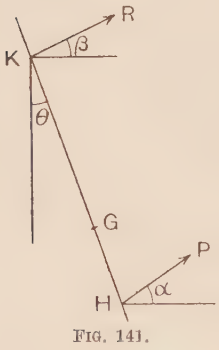

FIG. 141.

$$Mh(n' - n)\cos\theta = P\cos\alpha + R\cos\beta,$$
$$Mh(n' - n)\sin\theta = P\sin\alpha + R\sin\beta,$$

while (11) gives, by moments about K, if $KH = d$,

$$M(h^2 + k^2)(n' - n) = P.\,d\cos(\alpha - \theta).$$

These three equations determine n', R and β, being given the circumstances in which P is applied.

It follows that the components of R, resolved in the direction of GK and at right angles to it, are

$$R \sin(\beta-\theta) = -P \sin(\alpha-\theta),$$

and

$$R \cos(\beta-\theta) = -P \cos(\alpha-\theta) + Mh(n'-n)$$
$$= P \cos(\alpha-\theta)\left(\frac{hd}{h^2+k^2}-1\right).$$

The first is the direct response to the corresponding component of the blow and cannot vanish unless $\alpha=\theta$, when the blow is at right angles to GK. But the second will vanish when

$$d = KH = (h^2+k^2)/h.$$

But this condition identifies the point H with the centre of oscillation. Hence, when the pendulum receives a blow at the centre of oscillation in a direction at right angles to the line joining the centre of gravity to the centre of suspension, no impulsive reaction occurs at the axis about which the pendulum turns.

From this property, which is illustrated in many implements of industry and sport, the centre of oscillation has also been called the *centre of percussion*.

247. When the pendulum at the moment of receiving the blow is at rest in the vertical position of equilibrium, $n=\theta=0$. The previous equations then take the simple form:

$$R \cos\beta + P \cos\alpha = Mh \,.\, n',$$
$$R \sin\beta + P \sin\alpha = 0,$$
$$P \,.\, d \cos\alpha = M(h^2+k^2)n'.$$

But when the pendulum is set in motion, the equation of energy gives

$$\tfrac{1}{2}M(h^2+k^2)\dot\theta^2 = Mgh(\cos\theta - \cos\gamma),$$

where γ is the amplitude of the swing. In this case $\dot\theta = n'$ when $\theta=0$. Hence

$$(h^2+k^2)n'^2 = 2gh(1-\cos\gamma) = 4gh \sin^2\tfrac{1}{2}\gamma,$$

and therefore

$$P \cos\alpha = 2 \sin\tfrac{1}{2}\gamma \,.\, \frac{M}{d}\{gh(h^2+k^2)\}^{\frac{1}{2}}.$$

But the complete period of the pendulum, performing small oscillations freely, is

$$T = 2\pi\left(\frac{h^2+k^2}{gh}\right)^{\frac{1}{2}}.$$

Hence

$$P \cos\alpha = Mgh \,.\, T \sin\tfrac{1}{2}\gamma/\pi d,$$

and this equation expresses the horizontal component of P in terms of quantities which can be easily measured.

In the case of the *ballistic pendulum*, invented by Robins about 1740, a musket was attached horizontally to a heavy pendulum and fired, or else bullets were fired into the pendulum and remained imbedded. In either case the momentum communicated was $P\cos\alpha = mv$, and when m, the mass of the bullet, was measured, its velocity v could be deduced. In the second method the mass M must be taken to include the mass of the bullet lodged in the pendulum.

248. The property which characterizes the centre of percussion of a pendulum has a wider application. In an impulsive action ordinary forces play no sensible part, and in the case of the pendulum gravity and the force reacting on the axis at the instant do not enter into the results. Consider the application of an impulse P to a body at rest but free to move without any fixed axis. Let G be the centre of gravity of the body and H the foot of the perpendicular from G on the line of action of P. The body is assumed to be of such a form that, on receiving the blow, it will begin to turn about an axis perpendicular to the plane containing GH and P. In the line HG take a point K such that $KG=h$, $KH=d$ and

$$GH = d - h = h'.$$

Imagine a fixed axis passing through the point K and perpendicular to the plane of KGH and P. When the blow P is delivered, provided

$$hd = h^2 + k^2, \quad hh' = k^2,$$

that is, provided K and H have the relation of centre of suspension and centre of oscillation or percussion, no impulsive reaction will apply to the fixed axis which has been introduced. Hence this axis will have no effect and can be removed. In other words, K is the point about which the body will start to turn, even in the absence of any constraint compelling it to do so.

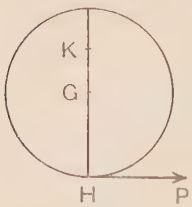

FIG. 142.

Ex. A uniform circular disc receives a tangential blow at a point on its edge. About what point will it begin to turn?

In this case $GH = h' = a$, the radius of the disc, and $k^2 = \frac{1}{2}a^2$. Hence $KG = h = k^2/h' = \frac{1}{2}a$. The disc will therefore begin to turn about the middle point of the radius opposite to that passing through the point at which the blow is delivered.

If P is the impulse, M the mass of the disc and n the initial angular velocity, by moments about the stationary point K,

$$(\tfrac{1}{2}Ma^2+\tfrac{1}{4}Ma^2)n=\tfrac{3}{2}aP, \qquad n=2P/Ma.$$

Were the disc constrained to turn about its centre G, its angular velocity n' would be given by $\tfrac{1}{2}Ma^2n'=aP$ or $n'=2P/Ma$, which is the same as when the disc is free. But its energy would be

$$\tfrac{1}{2}\cdot\tfrac{1}{2}Ma^2n'^2=P^2/M,$$

whereas the energy given to the disc, when free, is

$$\tfrac{1}{2}(\tfrac{1}{2}Ma^2+\tfrac{1}{4}Ma^2)n^2=\tfrac{3}{8}Ma^2n^2=\tfrac{3}{2}P^2/M,$$

or half as much again as when constrained.

249. The object of this chapter has been mainly to consider the motion in two dimensions of a single rigid body under given forces or impulses. The substance of § 216, and equation (3) of that section, will now be discussed more generally.

The term moment as applied to forces can clearly be extended to any localized vector. Let AB represent a vector associated with the point A, and $A'B'$ the corresponding vector associated with A', a subsequent position of A. Complete the parallelogram $A'B'BC$ and join AC, AB'. Then the vector $CA=CB-AB=A'B'-AB$, or CA represents the difference of the vectors $A'B'$, AB when transferred to A.

Let O be any fixed point, and join O to A, B, A', B' and C. The area of the triangle OAB represents half the moment of AB about O, and similarly other moments are represented by areas. Now

$$\begin{aligned} OA'B'-OAB&=OCB+\tfrac{1}{2}CA'B'B-(OCAB-OCA)\\ &=\tfrac{1}{2}CA'B'B-CAB+OCA\\ &=AA'B'+OCA. \end{aligned}$$

Hence, in words, the change in the moment of the vector AB about O is equal to the moment of the difference CA of the vectors about O, together with the moment of $A'B'$ about A. When these changes are divided by the corresponding time, and both the time and the displacement of the point A are indefinitely diminished, the rate of change may be substituted for the word change in this statement. The result of this process on the moment of $A'B'$ about A is less easy to interpret, but this part will simply vanish if the vector $A'B'$ coincides in direction with the displacement AA'.

Hence the result: the rate of change of the moment about a fixed point of a vector associated with a moving point is equal to

the moment of the rate of change of the vector about the same fixed point, provided the direction of the vector coincides with that of

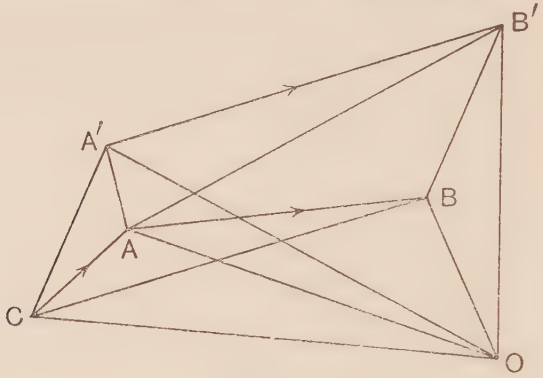

FIG. 143.

the velocity of the moving point. This can be proved very easily by differentiation, but the above method may be instructive.

250. The momentum of a particle, having the components $(m\dot{x}, m\dot{y})$, is clearly a vector which satisfies the condition by agreeing in direction with the velocity of the particle. The *moment of the momentum* is $m(x\dot{y} - y\dot{x})$, and the rate of change of the momentum is a vector of which the components are $\{d(m\dot{x})/dt, d(m\dot{y})dt\}$. Hence, in the most general form,

$$\frac{d}{dt}\{m(x\dot{y} - y\dot{x})\} = x\frac{d}{dt}(m\dot{y}) - y\frac{d}{dt}(m\dot{x}).$$

But the motion of m is defined by

$$\frac{d}{dt}(m\dot{x}) = X + R_x, \quad \frac{d}{dt}(m\dot{y}) = Y + R_y,$$

and from the sum of all such equations

$$\Sigma\left\{x\frac{d}{dt}(m\dot{y}) - y\frac{d}{dt}(m\dot{x})\right\} = \Sigma(xY - yX),$$

which is identical with (3) when m is constant, the reactions disappearing in the sum as already explained. But this can now be written

$$\frac{d}{dt}\Sigma m(x\dot{y} - y\dot{x}) = \Sigma(xY - yX),$$

or the rate of change of the moment of momentum of any material system about a fixed axis is equal to the moment of the impressed forces.

If no forces act on the system apart from the internal, self-compensating reactions, the rate of change of the moment of momentum vanishes, and the moment of momentum of the system is unchanged, or

$$\Sigma m(x\dot{y} - y\dot{x}) = \text{const.}$$

This moment, which in the case of a rigid body rotating with angular velocity n about the fixed axis becomes Mk^2n, is also called the *angular momentum* of the system. The property last proved is known as the conservation of angular momentum.

It will be found instructive to refer to the results obtained in §§ 83, 84, and to consider them in relation to the form in which those found above are expressed.

251. *Ex.* Two wheels rotate with angular velocities n_1, n_2 about a common smooth axle. Mutual friction causes them to seize and rotate together. Find the energy lost, the moments of inertia being I_1, I_2.

The axle being smooth, the angular momentum of the system is unaltered when the wheels rotate with the common angular velocity n. Hence

$$(I_1 + I_2)n = I_1n_1 + I_2n_2.$$

The loss of energy is therefore

$$\begin{aligned}
&\tfrac{1}{2}I_1n_1^2 + \tfrac{1}{2}I_2n_2^2 - \tfrac{1}{2}(I_1 + I_2)n^2 \\
&= \frac{1}{2(I_1 + I_2)}\{(I_1 + I_2)(I_1n_1^2 + I_2n_2^2) - (I_1n_1 + I_2n_2)^2\} \\
&= \frac{1}{2(I_1 + I_2)}\{I_2I_2(n_1^2 + n_2^2 - 2n_1n_2)\} \\
&= \frac{I_1I_2}{2(I_2 + I_2)}(n_1 - n_2)^2.
\end{aligned}$$

Let the wheels be uniform discs of 6 in. radius, the one, of 30 lbs., making 200 revolutions per minute, and the other, of 20 lbs., making 300 revolutions per minute. Then $n_1 = 200 \cdot 2\pi/60 = 20\pi/3$ and

$$n_2 = 300 \cdot 2\pi/60 = 10\pi.$$

Also $$I_1 = \tfrac{1}{2} \cdot 30 \cdot \tfrac{1}{4} = 15/4, \quad I_2 = \tfrac{1}{2} \cdot 20 \cdot \tfrac{1}{4} = 10/4.$$

Hence the loss of energy in this case is

$$\begin{aligned}
\frac{I_1I_2}{2(I_1 + I_2)}(n_1 - n_2)^2 &= \frac{150/16}{25/2} \cdot (1 - \tfrac{2}{3})^2\, 100\pi^2 \\
&= \tfrac{3}{4} \cdot \tfrac{1}{9} \cdot 100\pi^2 = 25\pi^2/3 \\
&= 82{\cdot}2 \text{ lb. ft.}^2/\text{sec.}^2
\end{aligned}$$

The units employed in this calculation are the pound, foot and second. The result in foot-pounds is therefore $82{\cdot}2/32{\cdot}2 = 2{\cdot}55$.

252. Since a single body has three degrees of freedom in two dimensions, the three equations (9), (10) and (12),

$$M(\bar{u}'-\bar{u})=P, \quad M(\bar{v}'-\bar{v})=Q,$$
$$Mk_1^2(n'-n)=x_1Q-y_1P,$$

show that the body can be brought from any one state of motion to any other by applying a single suitable impulse (P, Q) at a selected point (x_1, y_1), or rather along a selected line through this point. The centre of gravity is the (moving) origin of reference, and the axes have fixed directions.

Consider the motion as defined by the motion of the point (x_1, y_1) itself, with rotations at the rates n, n' about it. Let (u, v), (u', v') be its components of velocity before and after the impulse acts. Relative to the centre of gravity its velocity changes from $(-ny_1, nx_1)$ to $(-n'y_1, n'x_1)$. Hence

$$u=\bar{u}-ny_1, \quad v=\bar{v}+nx_1,$$
$$u'=\bar{u}'-n'y_1, \quad v'=\bar{v}'+n'x_1,$$

and therefore

$$P=M(u'-u)+M(n'-n)y_1,$$
$$Q=M(v'-v)-M(n'-n)x_1,$$

while $(n'-n)$ is given by

$$Mk_1^2(n'-n)=M\{x_1(v'-v)-y_1(u'-u)\}-M(x_1^2+y_1^2)(n'-n)$$

or

$$(k_1^2+r_1^2)(n'-n)=x_1(v'-v)-y_1(u'-u),$$

where $r_1^2=x_1^2+y_1^2$, so that $k_1^2+r_1^2$ is the square of the radius of gyration of the body about (x_1, y_1). When a point of a moving body is seized and made to move in a particular way, these equations determine the altered rotation and the impulse required at the point.

EXAMPLES.

1. The density of a circular disc, of radius a and mass M, varies as the distance from the centre. Prove that the moment of inertia about a diameter is $0{\cdot}3\,Ma^2$.

2. The density of a sphere of radius a varies as the distance from the centre. Prove that the radius of gyration of the sphere about a diameter is $\frac{2}{3}a$.

3. The radius of gyration of a straight thin rod about a perpendicular axis through its centre of gravity G is k. The mass of the rod, which may not be uniform, is M. Show that the rod is equi-momental with two particles placed at points A and B on it, such that $AG \,.\, GB=k^2$, and find their masses.

4. The sides of a parallelogram, of mass M, are a and b, meeting at the angle ϕ. Prove that the moment of inertia about either diagonal is

$$I = \frac{M}{6} \cdot \frac{a^2 b^2 \sin^2 \phi}{a^2 + b^2 \pm 2ab \cos \phi}.$$

5. A regular polygon of n sides, each equal to a, has the mass M. Prove that the moment of inertia about any side is

$$I = \frac{Ma^2}{48}\left(15 \cot^2 \frac{\pi}{n} + 1\right).$$

6. Three parallel lines meet a common perpendicular in the points A, B, C. If I_1, I_2, I_3 are the moments of inertia of a body about the three lines respectively, prove that the mass of the body is

$$M = \frac{I_1}{AB \,.\, AC} + \frac{I_2}{BA \,.\, BC} + \frac{I_3}{CA \,.\, CB},$$

with due regard to the signs of the segments.

7. Three lines in the plane of a lamina meet in the point O and make angles α, β, γ with a line OX. The moments of inertia of the lamina about them are I_1, I_2, I_3 respectively. Prove that the moment of inertia about a line through O making an angle ϕ with OX is

$$\frac{\sin(\phi - \beta)\sin(\phi - \gamma)}{\sin(\alpha - \beta)\sin(\alpha - \gamma)} I_1 + \frac{\sin(\phi - \gamma)\sin(\phi - \alpha)}{\sin(\beta - \gamma)\sin(\beta - \alpha)} I_2 + \frac{\sin(\phi - \alpha)\sin(\phi - \beta)}{\sin(\gamma - \alpha)\sin(\gamma - \beta)} I_3,$$

and deduce the moment of inertia of the lamina about the axis through O normal to its plane.

8. Prove that at the point where the side of a triangle is a principal axis of the area, the other principal axis bisects the line joining the middle point of the side to the opposite angle.

9. A circular disc is free to swing about a chord, which is fixed in a horizontal position. Find the length of the simple equivalent pendulum, and prove that this is least, and equal to the radius of the disc, when the chord is one side of an inscribed equilateral triangle.

10. A uniform right cone is suspended by its vertex. If the centre of oscillation is in the base, show that the height of the cone is equal to the radius of the base.

11. A rigid pendulum is drawn out of the vertical by a string connecting a point of the pendulum to a fixed point. When the string is suddenly cut and the pendulum released, show that in general the pressure on the suspension is simultaneously changed, and that if there is to be no change the line of the string must pass through the centre of oscillation at right angles to the axis of the pendulum.

12. A rigid pendulum oscillates about a horizontal axis. Prove that it passes through one position on either side of the vertical, such that the vertical pressure on the axis is then equal to the weight of the

pendulum, and that when the motion is just sufficient to give a complete turn this happens at the point where the inclination of the pendulum to the vertical is $\sec^{-1} 3$.

13. The weight of a symmetrical rigid pendulum is W, and its centre of gravity is at a depth h below the centre of suspension. The length of the simple equivalent pendulum is L, and a weight W' is attributed to the bob, such that $W'L = Wh$. If both pendulums are started with the same amplitude and phase, show that the pressure of the rigid pendulum on its axis will always differ from that of the simple pendulum on its support at the same instant by the constant vertical force $W - W'$.

14. A rough cylinder of radius a rotates on its axis, which is horizontal, with uniform angular acceleration Ω. A uniform cylinder is laid on the first, the plane of the two parallel axes making the angle θ with the horizon. Show that the second cylinder can maintain its position if $a\Omega = 2g \cos\theta$.

15. Two unequal weights are connected by a string passing over a heavy pulley on a smooth horizontal axis. If the pulley is a uniform cylinder, prove that the motion is the same as if the pulley's mass were neglected and each of the weights were increased by one quarter of the weight of the pulley.

16. A spindle in the form of two solid cones joined base to base is laid with its axis horizontal symmetrically across two parallel rails inclined at the angle α to the horizon. If the spindle rolls down without slipping on the rails, prove that the acceleration is

$$g \sin\alpha \cdot \frac{5(2h-b)^2}{26h^2 - 20bh + 5b^2},$$

where h is the height of each cone and b is the distance between the rails. Show also that the angular acceleration is greatest, the dimensions of the spindle being given, when

$$b = \tfrac{1}{5}h(10 - \sqrt{30}).$$

17. A uniform rod rests with one end on a smooth horizontal plane, and the other attached to a fixed point by a string of length l. Show that it will oscillate in a vertical plane in the same period as a simple pendulum of length $2l$.

18. A uniform rod of length $2c$ rests in equilibrium inside a smooth sphere of radius a. Find the time of oscillation when the rod is displaced in its vertical plane. Show that this time is least when the rod subtends a right angle at the centre if the radius of the sphere be given, or when the rod subtends an angle of $120°$ at the centre, if the length of the rod be given.

19. A uniform hemisphere of radius a stands with its curved surface on a horizontal plane. Prove that it will oscillate when disturbed in the same period as a simple pendulum of length $\frac{8}{15}a$ if the plane be rough, or of length $\frac{83}{120}a$ if the plane be smooth.

20. A uniform rod is suspended at its ends from two points on the same horizontal level by two equal vertical strings of length l. If the rod be turned slightly about the vertical line through its middle point, prove that it will oscillate in the same time as a simple pendulum of length $\frac{1}{3}l$.

21. A hollow cylinder is formed by fastening the two sides of a uniform sheet of metal, of mass M and width $2\pi a$, to a uniform rod of mass m. The cylinder is laid on a horizontal plane with the rod uppermost, and slightly disturbed. When the rod reaches its lowest position, prove that the square of the angular velocity of the cylinder is

$$\frac{4mg}{a} \cdot \frac{1}{2M}$$

when the plane is rough enough to prevent slipping, or

$$\frac{4mg}{a} \cdot \frac{M+m}{M(M+2m)}$$

when the plane is smooth.

22. A uniform sphere is poised on the highest point of a fixed rough sphere. When it rolls down, show that it will leave the fixed surface when the line joining the centres of the two spheres makes the angle $\sec^{-1} 1{\cdot}7$ with the vertical.

23. A moving mass has one degree of freedom, defined by the coordinate z, and the equation of energy can be found. Explain how the corresponding acceleration $\ddot{z}$ can be deduced.

A rigid body is pierced by a smooth vertical screw, of uniform pitch such that the body descends a distance h for one revolution. If k is the radius of gyration of the body about the axis of the screw, find the vertical acceleration when the body is allowed to fall.

24. A uniform rod standing on a rough horizontal plane is allowed to fall from rest in the vertical position. The lower end begins to slip when the rod makes the angle α with the vertical. Prove that the coefficient of friction on the plane is the positive value of

$$3 \sin \alpha (3 \cos \alpha - 2)(1 - 3 \cos \alpha)^{-2}.$$

In what conditions will the end of the rod slip backwards or forwards ?

25. A uniform beam rests on a smooth horizontal floor in contact with a smooth vertical wall. The lower end being slightly displaced, the beam falls down in a vertical plane. Prove that the upper end will leave the wall when it has descended a vertical distance equal to a third of the length of the beam, and that the pressure of the floor at the same instant is a quarter of the weight of the beam.

26. A lamina in the form of an equilateral triangle, lying on a smooth plane, receives a blow directed along a tangent to its inscribed circle. Show that the lamina will turn about the other end of the diameter of this circle through the point of contact.

27. A pendulum is struck a blow, and the impulsive reaction on the fixed axis is at right angles to the blow. Show that the point at which the blow is delivered must lie below the centre of percussion.

28. A lamina, of which the centre of gravity is G, turns in its own plane about the point A. It is then released at A and caused to turn about a point C. If the angular velocity is unchanged, show that the locus of C is the circle on AG as diameter.

29. A uniform rod of mass M and length $2a$, moving parallel to itself with velocity v at right angles to its length, strikes a stationary particle of mass m, which adheres to the rod, at a distance x from its middle point. Prove that the impulsive blow is

$$P = \frac{Mma^2v}{(M+m)a^2+3mx^2},$$

and that the loss of energy in the system is $\frac{1}{2}Pv$.

30. Prove that the angular momentum of a material system about an axis fixed in space is equal to the angular momentum about that axis of a particle of equal total mass coinciding with the centre of gravity G, together with the angular momentum of the system about a parallel axis through G.

31. A hoop of radius a is rolling in a vertical plane with angular velocity n on level ground when it meets directly a rough horizontal rail at a height h from the ground. Prove that if the hoop is to surmount the obstacle, and remain in contact with it until it has done so, the conditions

$$4gh < n^2(2a-h)^2 < 4g(a-h)$$

must be satisfied.

32. A uniform bar of length $2a$ rests symmetrically on two pegs at a distance $2c$ apart on a horizontal line. One end is raised and released, and the bar goes on striking each peg in turn. If there is no recoil or slipping at impact, and if $3c^2 < a^2$, prove that the angular velocity is reduced at each impact in the ratio a^2-3c^2 to a^2+3c^2.

PART V

CHAPTER IX

ELASTICITY

253. It is inevitable that in dealing with material bodies, which are the subject-matter of mechanics, we have used concepts which bear only an approximate resemblance to actual things. Thus a particle endowed with finite mass and yet possessing no extension in space can have no real existence. A light string or bar cannot really be without mass, though by neglecting the mass a useful solution to a practical problem may be obtained. Above all a truly rigid body, invariable in shape and volume under the application of stresses, does not and cannot exist, though the resistance to deformation may be extremely high. A massless inextensible string is a fiction in respect of both the attributes assigned to it.

It has been seen that the existence of an absolutely rigid material would involve the failure of a law of energy at a very early stage, as in the case of two equal rigid spheres meeting directly with equal opposite velocities. Both will be brought to rest, and as by the hypothesis of rigidity the spheres would suffer no change by the impact, the kinetic energy before impact will be simply destroyed without the possibility of compensation. What actually happens on the contrary is that the kinetic energy is first transformed into potential energy through the work of deformation done against elastic forces. The potential energy then reappears in its original form, but never more than partially, for there is always a balance which is converted during the process of impact into heat energy, which is nothing else than molecular kinetic energy.

Equally, if two equal masses connected by an inextensible

string are thrown apart in opposite directions the kinetic energy of the system will be destroyed when the string becomes taut. In fact the energy of a system can remain constant in spite of collisions only when the bodies admit of deformations involving work. Even so it is also necessary to take account of transformations into heat energy, and this is also true in the case of friction. Thus problems are raised which pass beyond the range of mechanics into the domain of thermodynamics. This extension of the subject need not arise when attention is confined to the conditions of equilibrium.

254. Consider a uniform elastic string stretched by two equal, opposed forces P applied at the two ends. If these are the only external or impressed forces the string will be in equilibrium. If now the string be cut at any point, equilibrium can only be maintained by applying to each part equal forces P at the point of section. Hence, while the string is still intact the same stress $\pm P$ must be acting at every point, and any part of the string, as much as the whole length, must be in equilibrium under the same forces P applied internally to its ends. The proportionate extension of the string will therefore be uniform along its length, and will depend on the magnitude of P and the nature of the string, but not on its length. Let l_0 be the natural length of the unstretched string, l its length under tension, and α the longitudinal *strain*. By this term is meant that unit length in the unstretched state becomes $1+\alpha$ when stretched, so that

$$\frac{1+\alpha}{1}=\frac{l}{l_0}, \quad \alpha=\frac{l-l_0}{l_0}.$$

Then the relation between P and α, so long as α does not exceed a certain limit, is expressed by

$$P=\lambda\alpha=\lambda(l-l_0)/l_0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where λ depends on the material and cross-section of the string, but not on its length. As λ is here multiplied by a ratio or pure number, its dimensions are evidently those of a force. This is the first illustration of Hooke's famous law : *ut tensio* (strain) *sic vis* (force).

255. Were m similar parallel strings taken together, or, what comes to the same thing, were the cross-section of the string multiplied by m, a force $Q=mP$ would be required to produce the same longitudinal strain α. Thus

$$Q=mP=m\lambda\alpha.$$

For a string of unit area we therefore write

$$T = E\alpha,$$

and for a string of cross-section m,

$$Q/m = T = E\alpha. \quad \text{.....(2)}$$

The stress is now measured by the force per unit area, and E is a constant depending only on the properties of the material and known as *Young's modulus*. Since

$$E = T/\alpha = Ql_0/m(l - l_0),$$

its dimensions are those of force per unit area or

$$MLT^{-2}/L^2 = ML^{-1}T^{-2}.$$

In the C.G.S. system the unit is therefore dyne/cm.^{2} or gr./cm. sec.^{2}

Ex. For a sample of steel wire $E = 2{\cdot}5 \times 10^{12}$ in C.G.S. units. To express the same modulus in tons' weight per sq. inch.

Since $g = 32{\cdot}2$ ft./sec.$^2 = 386$ in./sec.^{2},

and the unit of time is unaltered, the new measure is

$$\frac{2{\cdot}5 \times 10^{12}}{386} \cdot \frac{\text{gr.}}{\text{ton}} \cdot \frac{\text{in.}}{\text{cm.}}$$

$$= \frac{2{\cdot}5 \times 10^{12}}{386} \cdot \frac{1}{1{\cdot}016 \times 10^6} \cdot 2{\cdot}54$$

$$= 1{\cdot}62 \times 10^4 \text{ tons' weight per sq. in.}$$

Or, the change may be made by writing

$$E = \frac{2{\cdot}5 \times 10^{12}}{981} \text{ (gr. wt.)/cm.}^2$$

$$= 2{\cdot}5 \times 10^{12} \text{ (ton wt.)/in.}^2 \times \frac{(2{\cdot}54)^2}{981 \times 1{\cdot}016 \times 10^6}$$

$$= 2{\cdot}5 \times 10^{12} \text{ (ton wt.)/in.}^2 \times 6{\cdot}47 \times 10^{-9},$$

which leads to the same result. The factor of reduction is thus

$$6{\cdot}47 \times 10^{-9} = [1{\cdot}8110 - 10].$$

According to (2), under the stress $T = E$ the strain $\alpha = 1$ and the length of the wire is doubled. But it must be understood that long before this strain is reached the wire will break. The important limit to the stress (elastic limit) is that which allows the wire to recover its original state when the stress is removed. For the specimen in the above example this would be well under one per cent. of E. Large strains can be observed in rubber and similar substances.

When we are concerned only with the longitudinal stress and strain, as in the case of strings or wires, it will often be more convenient to use (1) $P = \lambda\alpha$, where P is the actual load, than (2) $T = E\alpha$, where T is

the strain per unit area. If m is the area of the cross-section, $P=mT$, and therefore $\lambda=mE$.

256. When a string is stretched from its natural length l_0 to l, work is done against elastic forces, and this is retained by the string in the form of potential energy. Since, within the limits of (1),

$$\frac{\lambda}{l_0}=\frac{P}{l-l_0}=\frac{P'}{l'-l_0},$$

where l' is an intermediate length and P' the corresponding tension producing equilibrium, such tensions may be set off as ordinates as

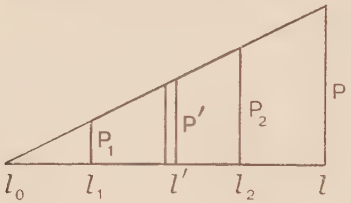

FIG. 144.

in the figure, with their extremities on a straight line. In a small extension under the tension P', the work done is the product of the small extension by the tension P', represented in the figure by a small quasi-rectangle of which P' is one side. The whole work done in the extension from l_0 to l is therefore represented by the sum of these rectangles, or the area of the triangle, so that it is

$$U=\tfrac{1}{2}P(l-l_0)=\tfrac{1}{2}\lambda(l-l_0)^2/l_0.$$

This is the potential energy stored in the string when stretched to the length l. Similarly the work done in stretching the string from the length l_1 to the length l_2 is

$$\tfrac{1}{2}\lambda\{(l_2-l_0)^2-(l_1-l_0)^2\}/l_0=\tfrac{1}{2}\lambda(l_2-l_1)(l_1+l_2-2l_0)/l_0,$$

or directly from the figure, as the area between the ordinates P_1, P_2,

$$\tfrac{1}{2}(P_1+P_2)(l_2-l_1),$$

which is sometimes a convenient form.

257. It has been seen that the tension is uniform along the string when the only external forces are equal opposite forces applied at the two ends. The mass of the string is then immaterial. When forces are applied at intermediate points the tension is no longer uniform.

Consider a heavy string hanging vertically. Let its natural length be l_0 and its weight per unit length be w. Let a weight W

be attached at the lower end. Consider the equilibrium at the point which in the unstretched state is a distance x below the point of suspension. The upward tension at this point must sustain the weight W and the weight of the lower section of the string, $w(l_0-x)$. Hence the tension at the point is

$$P=W+w(l_0-x),$$

which at the point of suspension ($x=0$) is $W+wl_0$ and at the lowest point ($x=l_0$) is W. Against the natural length of the string $AB=l_0$

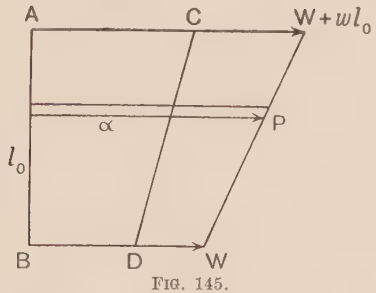

FIG. 145.

the tensions can be laid off horizontally as in the figure, the ends falling on a straight line. Another straight line CD is obtained by changing the lengths in the ratio of $1/\lambda$, so giving the longitudinal strain $\alpha=P/\lambda$. The extension of a small element of natural length is given by the product of the element by the corresponding value of α and is represented by a small quasi-rectangle. Hence the total extension of the string is represented by the sum of these rectangles or the area $ABDC$, so that it is

$$l-l_0=\tfrac{1}{2}l_0(AC+BD)=\tfrac{1}{2}l_0(2W+wl_0)/\lambda.$$

Were the string weightless ($w=0$) the extension would be Wl_0/λ, while if the weight W were removed, the extension would be $\frac{1}{2}wl_0^2/\lambda$. These two separate effects would be equal if $W=\frac{1}{2}wl_0=$half the weight of the string. It will therefore be seen that the total effect can be obtained by treating the string as weightless and adding half its weight at the lower end.

This rule will give the position of the lower end, but not of any point above it. At the unstretched distance x below the point of suspension, the weight $W+w(l_0-x)$ acts with full effect and must be increased by $\frac{1}{2}wx$ if the weight above is to be neglected. The fictitious tension is thus $W+w(l_0-\frac{1}{2}x)$, and the extension of the upper section of the string is

$$x'-x=x\{W+w(l_0-\tfrac{1}{2}x)\}/\lambda,$$

258. It is natural to approach the subject by considering first the elastic string under tension because the attention is here directed to a simple one-dimensional (longitudinal) stress and strain. The cross-section may be presumed small in comparison with the length, and the effect of strains in the lateral direction are relatively inconspicuous. As a subject for experiment the ordinary string is far from satisfactory because it is not at all uniform. On the other hand the simple theory has a very wide application. Not only does it apply directly to wires under tension, but also to cylinders under uniformly distributed load, for pressure is simply negative tension, and extension can be replaced by contraction. The limits of Hooke's law must be observed, and it will generally be more convenient to use equation (2), which involves Young's modulus E characteristic of the substance and the intensity of the stress per unit area.

Even within the limits of Hooke's law we now make a further simplification by assuming that the strains are so small that their squares and products can be neglected. Thus if α and β are two strains superposed,

$$(1+\alpha)(1+\beta)=1+\alpha+\beta,$$

the product $\alpha\beta$ being neglected. Hence the effects of strains can be simply added, each being calculated as if it alone existed, acting as though the material were otherwise unstrained.

259. As a string or bar of uniform section is a natural subject for the study of longitudinal strain, so a membrane or plate of uniform thickness is the natural subject for the study of strains in two dimensions. The membrane, like the string, may be considered capable of resisting tensions only, while the plate, like the bar, is capable of resisting tensions or pressures, the difference being merely one of sign. For the present all the forces are parallel to the surfaces of the plate, and strains in the direction normal to the surface are not of immediate interest. As the forces are represented by stresses per unit area normal to the surface of the plate, the thickness of the plate is immaterial.

Opposed equal forces impressed uniformly on the plate will establish a uniform tension in one direction, as in the case of a string. Consider a square $ABCD$ of unit area, with its sides AD, BC parallel to the stress T. We are now concerned with relative displacements alone and A may be treated as a fixed point. Then the effect of the simple stress T is to strain the square $ABCD$ into

the rectangle $AB'C'D'$. For the extension in the direction AD the same law holds as for the string, namely,

$$DD'/AD = \alpha = T/E,$$

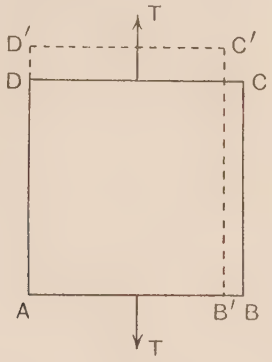

FIG. 146.

where E is Young's modulus. But this direct strain is accompanied by an indirect strain, a contraction in the direction AB, expressed by

$$BB'/AB = \beta = \sigma T/E,$$

where the fraction σ is a constant characteristic of the material of the plate. This constant, which is known as *Poisson's ratio*, is clearly a pure number, with no physical dimensions.

260. The effect of two uniform stresses crossed at right angles

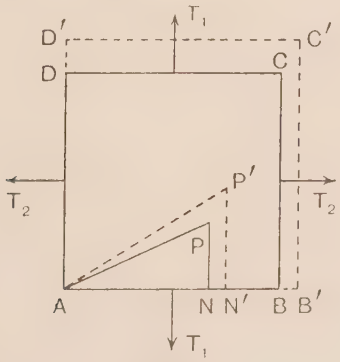

FIG. 147.

can now be seen. In the direction AD the direct strain due to T_1 is $\alpha_1 = T_1/E$ and the indirect strain due to T_2 is $-\beta_2 = -\sigma T_2/E$.

Similarly in the direction AB the direct strain due to T_2 is $\alpha_2 = T_2/E$ and the indirect strain due to T_1 is $-\beta_1 = -\sigma T_1/E$.

Hence the square $ABCD$ is strained into the rectangle $AB'C'D'$, where

$$DD'/AD=\alpha_1-\beta_2=T_1/E-\sigma T_2/E,$$

and

$$BB'/AB=\alpha_2-\beta_1=T_2/E-\sigma T_1/E.$$

Consider a point P displaced by the strain to P'. Let $AP=r$, $PAB=\theta$, $AP'=r'$, $P'AB'=\theta'$. Then owing to the uniformity of the stresses,

$$\frac{r'\cos\theta'}{r\cos\theta}-1=\frac{AN'}{AN}-1=\frac{NN'}{AN}$$
$$=\frac{BB'}{AB}=\frac{T_2-\sigma T_1}{E},$$

and similarly, in the perpendicular direction,

$$\frac{r'\sin\theta'}{r\sin\theta}-1=\frac{DD'}{AD}=\frac{T_1-\sigma T_2}{E}.$$

Hence

$$\frac{\tan\theta'}{\tan\theta}=\left(1+\frac{T_1-\sigma T_2}{E}\right)\Big/\left(1+\frac{T_2-\sigma T_1}{E}\right),$$

which shows that θ' is independent of r. Therefore every point on the straight line AP after the strain lies on the straight line AP', of which the direction bears a definite relation to the direction of AP. As the condition of stress is the same at all points it follows that after strain straight lines remain straight lines and parallel lines remain parallel. These are the characteristic properties of a *homogeneous strain.*

It will be noticed that if $T_1=T_2$, $\theta'=\theta$ and every straight line remains parallel to itself after strain. Otherwise if T_1 and T_2 are unequal, θ' and θ are unequal and the line is rotated through an angle, except in the special directions $\theta=0°$ (180°) or 90° (270°) parallel to the crossed stresses.

261. Corresponding to the crossed stresses we now find the action across any line CE making the angle θ with CD, or, more properly, across the section of the plate of which CE is the trace. This action may be resolved into a normal tension P and a traction Q along CE. The principal stresses will now be taken as $T_1=T+T'$ and $T_2=T-T'$, where $T=\frac{1}{2}(T_1+T_2)$, $T'=\frac{1}{2}(T_1-T_2)$. The triangle CED is in equilibrium under the forces acting on its sides, and the relations obtained by resolving parallel to P and Q are (suppressing the thickness, which is a constant factor)

$$P\,.\,EC=(T-T')\,DE\,.\sin\theta+(T+T')\,CD\cos\theta,$$
$$Q\,.\,EC=(T+T')\,DC\,.\sin\theta-(T-T')\,DE\,.\cos\theta,$$

or $$P=(T-T')\sin^2\theta+(T+T')\cos^2\theta=T+T'\cos 2\theta,$$
$$Q=(T+T')\sin\theta\cos\theta-(T-T')\sin\theta\cos\theta=T'\sin 2\theta.$$

These relations are easily interpreted. The mean of the crossed stresses, $T=\frac{1}{2}(T_1+T_2)$, applies equally across a section of the plate

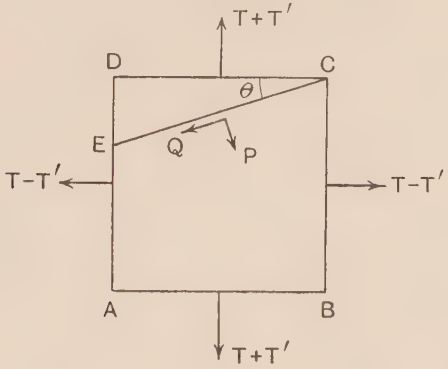

FIG. 148.

drawn in any direction. The corresponding strain is clearly a uniform expansion in all directions. When this constituent has been removed from the crossed stresses and from P, the remaining constituents can be interpreted graphically.

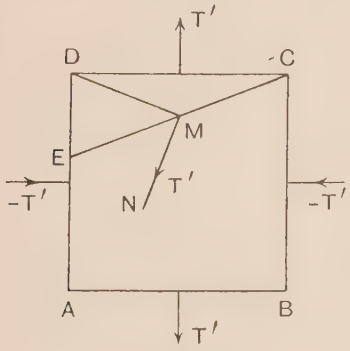

FIG. 149*a*.

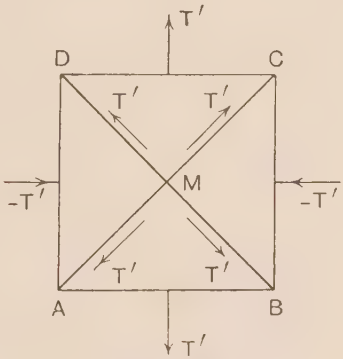

FIG. 149*b*.

Let M be the middle point of EC, or the centre of the circle EDC since EDC is a right angle. Join DM and draw MN to represent T' at right angles to DM. Then the angle $DME=2DCE=2\theta$, and the angle $EMN=90°-2\theta$. Hence the components of T' in this position MN are $T'\sin 2\theta=Q$ in the direction CE and $T'\cos 2\theta=P-T$ in the perpendicular direction. The action across CE resulting from the crossed stresses is therefore the oblique stress T' in the direction

MN as constructed, together with the normal stress T which is common to all directions.

When CE coincides with the diagonal CA of the square ($\theta = 45°$), $DMA = 90°$ and MN also coincides with CA. The oblique stress T' in this case reduces to a simple traction along the diagonal. The stresses along the diagonals are as in the figure.

262. It will now be convenient to consider the effects of the stresses T and T' with the above meanings separately. These effects can afterwards be very simply combined, if the stresses co-exist, within the limits of the principle of super-position.

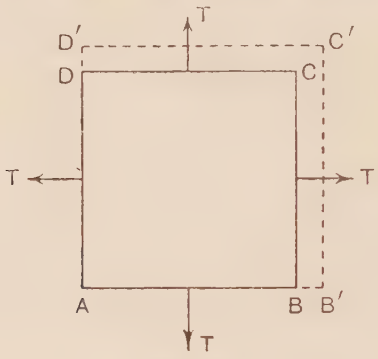

FIG. 150.

To begin with the uniform expansion due to the equal crossed stresses T. The square $ABCD$ is strained into the square $AB'C'D'$. In either direction, AB or AD, there is a direct strain T/E and an indirect strain $-\sigma T/E$. Hence the linear expansion is

$$\alpha = BB'/AB = DD'/AD = T/E - \sigma T/E.$$

The increase of area per unit area is $(1+\alpha)^2 - 1 = 2\alpha$, since α^2 must be neglected. This being the areal strain, corresponding to the uniform plane stress T, it would be quite feasible to define an areal modulus b as the ratio of the stress to the areal strain, so that

$$b = T/2\alpha = E/2(1-\sigma).$$

This, however, is not usual. Besides the longitudinal stress with which Young's modulus E is connected, it is easy in practice to apply a uniform (hydrostatic) stress in three dimensions, but not in two.

To see the effect of such a stress, let $ABCD$ of the previous figure be one face of a cube, a normal stress T being applied in

addition to this and the opposite face. This will cause a further indirect strain $-\sigma T/E$ in each of the sides of the square face $ABCD$, just as the same stress over the other faces through AB, CD causes this indirect strain in AB. In the uniform dilatation resulting from the uniform three-dimensional stress the strain in every side of the cube is therefore

$$\alpha = BB'/AB = T/E - 2\sigma T/E.$$

The increase of volume per unit volume is now $(1+\alpha)^3 - 1 = 3\alpha$, when α^2 and α^3 are neglected. The *bulk-modulus* k is defined as the ratio of T to this volume strain, and accordingly

$$k = T/3\alpha = E/3(1-2\sigma). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The dimensions of k are the same as those of E, namely $ML^{-1}T^{-2}$. This modulus retains a meaning when applied to fluids, being the reciprocal of the compressibility. Since E and k are positive, $\sigma < \frac{1}{2}$.

263. We now return to the consideration of a plate under the equal and opposite crossed stresses T', which, it has been seen, are also equivalent to a diagonal system of tractions T'. The square $ABCD$ is strained into the rectangle $A'B'C'D'$, the centre O being taken as the fixed point of reference. The middle points of the

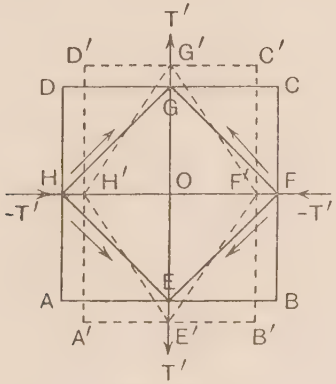

Fig. 151.

sides, E, F, G, H are displaced by the strain to E', F', G', H', the effect being to change the square $EFGH$ under the influence of tractions T' along its sides into the rhombus $E'F'G'H'$.

The strain in AB due to the normal stress $-T'$ on BC, AD, is

$-T'/E$ and that due indirectly to the normal stress T' on AB, CD is $-\sigma T'/E$. Hence the strain in AB or CD is

$$-\frac{F'F}{OF}=-\frac{HH'}{OH}=-\frac{T'}{E}-\frac{\sigma T'}{E}=-\alpha.$$

Similarly the strain in AD is T'/E directly from the normal stress T' and $\sigma T'/E$ indirectly from the normal stress $-T'$, or together

$$\frac{EE'}{OE}=\frac{GG'}{OG}=\frac{T'}{E}+\frac{\sigma T'}{E}=\alpha.$$

But since straight lines remain straight lines after homogeneous strain, the square $EFGH$ becomes by symmetry the rhombus $E'F'G'H'$, and this change may be connected independently with the system of tractions along its sides. From this point of view the strain is called a *shear*.

264. The following properties of a shear are easily deduced :

(1) The sides of the square $EFGH$ are unaltered by the strain. For if $OE=OF=1$, $OE'=1+\alpha$, $OF'=1-\alpha$. Hence

$$E'F'^2=(1+\alpha)^2+(1-\alpha)^2=2+2\alpha^2$$
$$=2=EF^2,$$

since α^2 is to be neglected. Thus $E'F'=EF$.

(2) The area of the square $EFGH$ is unaltered by the strain. For

$$\frac{E'F'G'H'}{EFGH}=\frac{\frac{1}{2}A'B'C'D'}{\frac{1}{2}ABCD}=(1+\alpha)(1-\alpha)$$
$$=1-\alpha^2=1$$

to the order adopted. Hence $E'F'G'H'=EFGH$ in area. And generally the effect of a simple shear is change of shape without change of size, while the effect of equal crossed normal stresses is change of size without change of shape.

(3) The thickness of the plate is unaffected. For as in the case of the cube previously considered, the indirect strain in this direction due to the normal stress $+T'$ is $-\sigma T'/E$ and the corresponding strain due to the normal stress $-T'$ is $+\sigma T'/E$. These two strains neutralize one another, and the thickness of the plate is therefore unaltered.

(4) The only effect of the shear being thus change of shape, this is naturally measured by the departure of the angles of the rhombus $E'F'G'H'$ from right angles. These angles being denoted by $90°\pm\theta$, θ is called the *angle of shear*. Thus $OE'F'=45°-\frac{1}{2}\theta$, and so

$$\tan(45°-\tfrac{1}{2}\theta)=\frac{OF'}{OE'}=\frac{1-\alpha}{1+\alpha}=\frac{1-\tan\frac{1}{2}\theta}{1+\tan\frac{1}{2}\theta}.$$

Hence $\tan \frac{1}{2}\theta = \frac{1}{2}\theta = \alpha,$

or $\theta = 2\alpha = 2(1+\sigma)T'/E.$

Now the small angle θ (in circular measure) represents the distortion due to the tractions T' acting on the sides of the square. These are proportional, another illustration of Hooke's law, and a new modulus is defined by the ratio. This is the *rigidity* n, and thus

$$n = T'/\theta = E/2(1+\sigma). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Recalling equation (3),

$$k = E/3(1-2\sigma),$$

we have four elastic constants connected by two relations. If n and k are taken as fundamental,

$$\frac{3k}{n} = \frac{2+2\sigma}{1-2\sigma}, \quad \text{or} \quad 2\sigma = \frac{3k-2n}{3k+n}, \quad 2\sigma+2 = \frac{9k}{3k+n}.$$

Hence $$E = \frac{9nk}{3k+n}, \quad \sigma = \frac{3k-2n}{2(3k+n)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

express Young's modulus E and Poisson's ratio σ in terms of n and k. As the circular measure θ is a pure number, like any other strain, it is clear that the dimensions of n are the same as those of E or k.

265. The work done in a shear can be calculated in two ways. Let t be the thickness of the plate, and consider a unit square under the normal stress $\pm T'$. The extension α in one direction under the forces $T't$ involves work, as in the case of the string, $\frac{1}{2}T't\alpha$. The contraction $-\alpha$ in the other direction under the forces $-T't$ also involves the work $\frac{1}{2}T't\alpha$. In all the work done is therefore $T't\alpha$ per unit area of the plate, or $T'\alpha$ per unit volume.

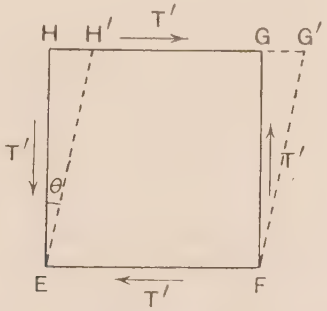

FIG. 152.

But the shear can be considered more simply by considering the unit square under tractions along its sides. Since relative displace-

ments are alone important and the square $EFGH$ is strained into a rhombus $E'F'G'H'$ with equal sides, the side EF may be taken as fixed and GH displaced to $G'H'$ in its own line. Being between the same parallels the equality of $EFGH$ and $EFG'H'$ in area is clear, but it may seem that $EH'>EH$. The answer is that

$$EH=EH'\cos\theta,$$

and θ, the angle of shear, being small, $\cos\theta=1$ to the second order which is neglected. In the displacements HH' and GG' the forces $T't$ along the perpendicular directions EH, FG do no work. Only the force $T't$ along HG does effective work in the strain, and since this force grows from zero, as in the gradual extension of a string, the work done per unit area of the plate is

$$\tfrac{1}{2}T't\,.\,HH'=\tfrac{1}{2}T't\,.\,\tan\theta=\tfrac{1}{2}T't\,.\,\theta=T'ta$$

as before. Since $T'=n\theta$, the work done per unit volume is

$$\tfrac{1}{2}T'\theta=\tfrac{1}{2}n\theta^2.$$

266. The rigidity n can be most simply determined by twisting a wire. But first consider a hollow cylinder of length l, radius a and small thickness t.

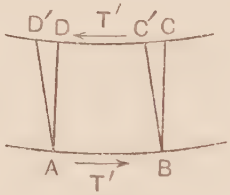

FIG. 153.

The lower edge of the cylinder being divided into small intervals AB, small squares $ABCD$ may be drawn on the surface of the cylinder. A twist will be produced by a traction T' round the lower edge and an opposite traction T' round the circle CD. This will cause a shear of angle θ, changing $ABCD$ into $ABC'D'$, and $T'=n\theta$.

The tangential force on AB is $T'.AB.t$ and varies in direction round the edge, but the moment of this force about the axis is $aT't\,.\,AB$, and in the sum, for all the square elements round the edge, $aT't\,.\,\Sigma AB=2\pi a^2T't=2\pi na^2\theta t$. This circumferential torque is communicated from ring to ring along the cylinder, and for equilibrium must be balanced by an equal and opposite torque on the upper edge.

The relative displacement in one layer is $CC' = BC \tan\theta$, and therefore for the lower edge relative to the upper

$$\Sigma BC \,.\, \tan\theta = l \tan\theta.$$

On the other hand, if ϕ is the total twist of the lower edge relative to the upper, this displacement is $a\phi$. Hence

$$a\phi = l \tan\theta = l\theta,$$

for ϕ will not be large, and θ will be much smaller if l is large in comparison with a. Hence the torque required to produce the twist ϕ will be

$$N = 2\pi n a^2 \theta t = 2\pi n a^3 t \,.\, \phi/l,$$

and, as simple shears in the surface of the cylinder alone are involved, the thickness of the cylinder will be unaffected.

267. The torque required to twist a solid cylinder or wire can now be deduced, when one plane end is twisted through an angle ϕ relative to the other. The wire may be supposed to be divided into a large number of coaxal thin cylinders, of radii $r_1, r_2, \ldots, r_n, r_{n+1}, \ldots$, and of equal thickness. The torque required by the cylinder between radii r_n and r_{n+1} is $2\pi n r^3 (r_{n+1} - r_n) \,.\, \phi/l$, where r is intermediate between r_n and r_{n+1}. Hence the torque required by the solid cylinder of radius a is

$$\begin{aligned} N &= 2\pi n \,.\, \Sigma r^3 (r_{n+1} - r_n) \,.\, \phi/l \\ &= \tfrac{1}{2}\pi n \Sigma (r_{n+1}{}^4 - r_n{}^4) \,.\, \phi/l \\ &= \tfrac{1}{2}\pi n a^4 \,.\, \phi/l, \end{aligned}$$

being applied in opposite senses in the planes of the two ends.

If, the upper end of the wire being held fixed, the lower end is loaded with a mass of moment of inertia Mk^2 and released in the position ϕ, the elastic forces will have a moment $-N$. Hence, $\ddot{\phi}$ being the angular acceleration,

$$Mk^2 \ddot{\phi} = -\phi \,.\, \pi n a^4/2l,$$

and by analogy with other equations of this form the period of vibration will be

$$P = 2\pi \sqrt{\frac{2Mk^2 l}{\pi n a^4}} = \sqrt{\frac{8\pi Mk^2 l}{n a^4}},$$

whence

$$n = \frac{8\pi Mk^2 l}{a^4 P^2},$$

and thus n can be determined.

268. When a bar is bent the conditions are more complicated. The outer part is expanded, the inner contracted, and the two are

separated by a neutral surface which is free from longitudinal strain. Take the case of a rod, of uniform square section of side a, bent in a

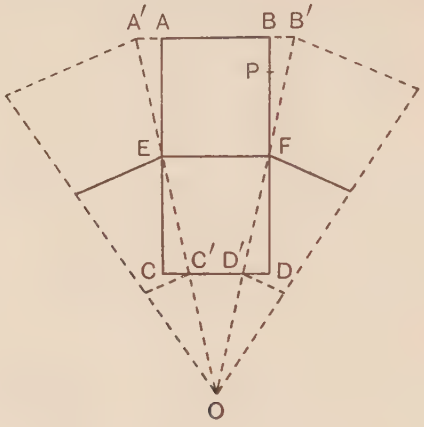

FIG. 154.

plane parallel to a side. The figure represents the plane of bending through the axis of the rod, of which EF is a part. $AECDFB$ is a thin section of the rod before bending, and this is deformed into $A'EC'D'FB'$. By considering similar neighbouring sections it is evident that $A'B'$, $C'D'$ and the neutral axis EF are parts of continuous curves (circles). If $A'C'$, $B'D'$ meet in O, $OE=OF=\rho$ is the radius of the neutral axis. With slight curvature O is a distant point and $EOF=\theta$ is a small angle. Thus $AB=CD=EF=\rho\theta$, while $A'B'=(\rho+\frac{1}{2}a)\theta$ and $C'D'=(\rho-\frac{1}{2}a)\theta$. The strain in $A'B'$ is therefore $(A'B'-AB)/AB=a/2\rho$ and the strain in $C'D'$ is

$$(C'D'-CD)/CD=-a/2\rho.$$

The corresponding stresses normal to AC or BD, required to produce these strains, are $aE/2\rho$ at A or B, 0 at E or F and $-aE/2\rho$ at C or D, where E is Young's modulus. Moreover, at an intermediate point such as P, where $FP=x$, the strain, being proportional to FP, is x/ρ and the corresponding stress is xE/ρ. The symmetry with opposite sign of these forces on BD with respect to F shows that they are equivalent to a couple, of which the moment about an axis through F can now be calculated.

269. Let $JKLR$ be a cross-section of the rod, the diameter BFD being in the plane of bending and the diameter GFH being the trace of the neutral surface. Let m be a small square element of surface, of which the position is defined by the coordinates $FP=x$, $FQ=y$.

The normal stress on m, as at P, is xE/ρ and the actual force on m is mxE/ρ. Its moment about the axis GFH is therefore mx^2E/ρ and the total moment of all the forces on the area $JKLR$ is

$$(\Sigma mx^2)E/\rho.$$

But Σmx^2, the sum being taken over all the elements which make up the area $JKLR$, is the moment of inertia of the square about its

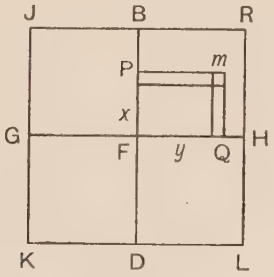

FIG. 155.

diameter GFH, and this is Ak^2, where $A=a^2$ is the area of the square and $k^2=a^2/12$. Hence $\Sigma mx^2=a^4/12$, and the moment of the couple, to which the forces on the section clearly reduce in this case, is $a^4E/12\rho$. The axis of the couple is GFH, which is in the neutral surface and perpendicular to the plane of bending.

If equal and opposite couples, of moment C, are thus applied at the two ends of the rod, the rod will be bent with uniform curvature of radius ρ, where C and ρ are connected by the relation

$$12\rho C=a^4E,$$

and the curve is a circle.

270. Every cross-section $JKLR$ will be slightly distorted from its square shape by the bending of the rod. For the normal stress $\frac{1}{2}aE/\rho$ near the edge JR will produce an indirect strain $-\frac{1}{2}a\sigma/\rho$ in JR, while the normal stress $-\frac{1}{2}aE/\rho$ near the edge KL will produce an indirect strain $\frac{1}{2}a\sigma/\rho$ in KL.

The neutral surface GH is unstrained. Thus the square section $JKLR$ is distorted into $J'K'L'R'$, and the lines $K'GJ'$, $L'HR'$ meet in a distant point O', which may be regarded as the centre of curvature of the trace GH.

This figure in the plane of the cross-section may be compared with the earlier figure in the perpendicular plane of bending, and as in that figure the strains $\pm a/2\rho$ in AB, CD correspond with a radius of curvature ρ in EF, so here the strains $\pm a\sigma/2\rho$ in KL, JR corres-

pond with a radius of curvature $-\rho' = \rho/\sigma$ in GH. Opposite signs are attached to ρ and ρ' because if the curvature of EF, the trace

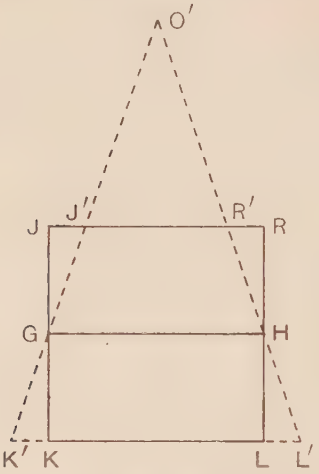

FIG. 156.

of the neutral surface in the plane of bending, is downwards, the curvature of GH, the trace of the neutral surface in a cross-section, is upwards.

The reasoning assumes that the sides JK, RL remain straight after the bending, and that the sides JR, KL become curved. The measure of the curvature of GH then follows. The assumption can be justified, but not by the elementary methods adopted here.

271. It has been seen that a uniform rod of square section can be bent in a plane by applying a simple couple suitably at each end.

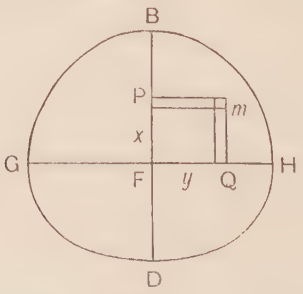

FIG. 157.

If now the section is of some other shape, the rod can be similarly bent, but only under certain conditions.

Let $BGDH$ be a cross-section, no longer square, GFH the trace of the neutral surface, BFD the trace of the plane of bending. As before these are at right angles, and the normal stress at P is xE/ρ. On a small square element m, of which the position is defined by $FP=x$, $FQ=y$, the stress is equally xE/ρ and the actual force is mxE/ρ. The resultant of all such parallel forces is to be a couple, and the single force resultant, $(\Sigma mx)E/\rho$, must vanish. Hence $\Sigma mx=0$, and this means that the centre of gravity of the cross-section lies on GFH, the trace of the neutral surface. This defines the position of the neutral surface, and the point F may be identified with the centre of gravity of the section. The forces now reduce to a couple, of which GFH must be the axis. The moment of all the forces on the section about BFD is $(\Sigma mxy)E/\rho$, and this must vanish, or there will be no *plane* of bending. Hence $\Sigma mxy=0$, and therefore FH, FB must be principal axes of the section. In general it is only about one of these two lines that the rod can bend so as to remain in a plane without additional constraining forces. Finally, when this condition is satisfied, and GFH is a principal axis of inertia through the centre of gravity of the section, this line is also the axis of the resulting stress couple, and the moment of the couple is

$$C=(\Sigma mx^2)E/\rho=Ak^2E/\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Here A is the area of the section of the rod, Ak^2 its moment of inertia about GFH, the line through the centre of gravity of the area perpendicular to the plane of bending, and ρ is the radius of curvature of the central axis of the bent rod, which is the locus of the centre of gravity F of every cross-section.

272. Uniform bending of a rod, as considered above, requires not only that the forces applied to the two ends should reduce to equal and opposite couples about certain axes, but that they should be distributed over the ends in a certain way. Yet the same approximate result is obtained when the couples are applied in any manner, as when a flexible rod is held casually in the hands. Any deviation so caused is confined mainly to the parts of the rod quite close to the ends.

In general the forces do not reduce to simple couples, but also include tangential stresses over a section. Even so, at a point distant from an end, the bending moment arising from such stresses involves the distance from the end as a factor, while the moment of

the local couple involves the width of the rod as the corresponding factor. The forces contributing to the couple are therefore much greater than the tangential stresses so long as the rod is thin and the point is not near the end of the rod. In the case of a loaded thin rod it is the normal stresses which are predominant, and the tangential stresses can be neglected.

273. Let AB be a thin rod in a horizontal position, loaded with a weight W at the end B. For the equilibrium of the length PB the

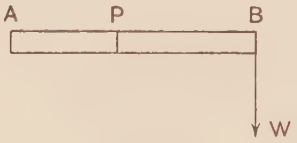

FIG. 158.

moment of the couple acting on the cross-section at P must be equal and opposite to $PB \,.\, W$. But if the curvature of the rod is very slight, and the tangential stresses are neglected, this moment is Ak^2E/ρ, where ρ is the radius of curvature of the central axis of the rod at P. Hence

$$Ak^2E/\rho = PB \,.\, W.$$

The curvature must be slight, or PB will not be approximately horizontal.

Let the end A be clamped horizontally, while the free end B supports the load W. Let AX be the horizontal axis and $ACY = 2\rho$

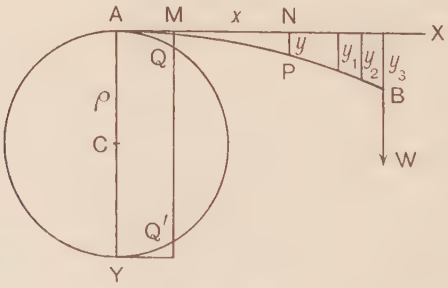

FIG. 159.

the vertical diameter of the circle of curvature at A. The position of any point P on the rod is defined by the abscissa $AN = x$ and the ordinate $PN = y$. Since the rod is to be considered very slightly bent, y is always small, and lengths x measured along AX are

indistinguishable from corresponding lengths AP measured along the central axis of the rod.

If MQQ' is parallel to AY, $AM^2 = MQ \,.\, MQ' = MQ(AY - MQ)$, or if $AM = x$, $MQ = y$ (cf. § 20),

$$x^2 = y(2\rho - y).$$

So long as Q on the circle of curvature is very close to A, y is very small and approximately equal to $x^2/2\rho$, while $y^2 = x^4/4\rho^2$ approximately is negligible. Now close to A the curve of the rod coincides with the circle of curvature, and a first approximation to the relation between x and y at P will naturally be of the form

$$y = x^2/2\rho - bx^3,$$

where the coefficient b is to be determined. As x increases the term bx^3 will be at first negligible, but will become appreciable before y^2 or x^4. It is therefore suitable for a first approximation.

274. As P approaches B the bending moment $PB \,.\, W$ tends to zero, and with it the curvature of the rod. In the immediate neighbourhood of B the rod is therefore straight, and if y_1, y_2, y_3 are three close equidistant ordinates, $y_1 + y_3 = 2y_2$.

Let the corresponding abscissae be $x - h$, x, $x + h$. Hence

$$(x-h)^2/2\rho - b(x-h)^3 + (x+h)^2/2\rho - b(x+h)^3 = 2(x^2/2\rho - bx^3),$$

or
$$(x^2 + h^2)/2\rho - b(x^3 + 3h^2x) = x^2/2\rho - bx^3,$$

or
$$h^2/2\rho - 3bh^2x = 0.$$

At B, $x = l$, the length of the rod, and the condition is satisfied, without making h absolutely vanish, if $1 = 6\rho bl$.

Hence the approximate relation between x and y along the rod is

$$y = x^2/2\rho - x^3/6\rho l$$
$$= x^2(3l - x)/6\rho l,$$

where ρ is the radius of curvature at A. But the bending moment here is

$$Wl = Ak^2E/\rho.$$

The relation may therefore be written

$$y = x^2(3l - x)W/6Ak^2E.$$

This gives the depression y below the level at A for any point of the rod. In particular, at the other end B, $x = l$ and the depression is

$$y_1 = Wl^3/3Ak^2E. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

At the point P or (x, y) on the rod the inclination of the tangent to the horizontal is given by

$$\tan\theta = dy/dx = (6lx - 3x^2)W/6Ak^2E,$$

so that

$$\theta = \tan\theta = x(2l-x)W/2Ak^2E,$$

for θ must be small. In particular, at the end B where $x=l$,

$$\theta = Wl^2/2Ak^2E = \tfrac{3}{2}y_1/l,$$

where y_1 is the vertical depression of B below A. The amount of bending between P at (x, y) and B is thus

$$\begin{aligned}\phi &= W\{l^2 - x(2l-x)\}/2Ak^2E \\ &= W(l-x)^2/2Ak^2E,\end{aligned}$$

which varies as $(l-x)^2$ or PB^2.

275. As an example, consider a thin beam of length l supported horizontally at its ends A and B with a load W applied at the middle point C.

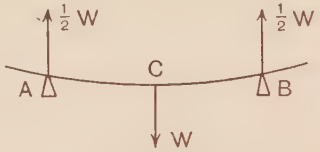

FIG. 160.

By symmetry the beam must be really horizontal at C and for equilibrium the supports must apply forces $\frac{1}{2}W$ upwards at A and B. Hence the conditions for either half of the beam are the same as if the beam were clamped horizontally at C and the upward force $\frac{1}{2}W$ were applied at A and B. The elevation of A or B above C is therefore obtained by substituting $\frac{1}{2}W$ for W and $\frac{1}{2}l$ for l in the equation (7) for a rod clamped at one end with the other end loaded and free, and it is

$$\begin{aligned}y &= \tfrac{1}{2}W(\tfrac{1}{2}l)^3/3Ak^2E \\ &= Wl^3/48Ak^2E.\end{aligned}$$

This gives a simple practical method of determining Young's modulus E for the beam. It will be noticed that in this simple theory the weight of the beam itself is neglected, and that its distribution along the length would introduce some complexity into the problem. Yet the practical value of the result is not disturbed. Within the limits of the principle of superposition of strains the depression of C below AB is the sum of the depressions due to the beam itself and to the load, and the depression y to which the formula refers must be measured relatively to the axis of the beam

supported without load, and not relatively to the true horizontal line AB.

276. As a second example, take a thin beam AB of length l, clamped horizontally at both ends. The function of the clamp is to impose direction as well as support, without applying longitudinal force, and may be represented by a short smooth tube through which the beam passes.

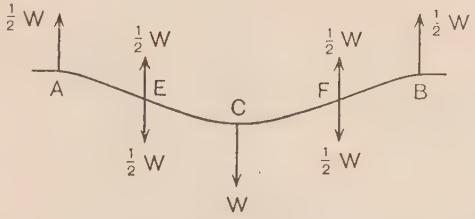

FIG. 161.

As before, the support implies an upward force $\frac{1}{2}W$ at A and B when a load W is applied at the middle point C. It is evident that the curvature will be downwards at A and B, and upwards at C, while at some intermediate points E, F, the curvature changes sign and vanishes. The couples at E and F must vanish too. Equilibrium of the portion AE requires a downward force $\frac{1}{2}W$ at E to balance the upward force $\frac{1}{2}W$ at A, and a couple $\frac{1}{2}W.AE$ at A. The upward reaction $\frac{1}{2}W$ at E balances the downward force $\frac{1}{2}W$ at C, and similar conditions apply to the portions CF, FB.

Hence each of the four portions AE, CE, CF, BF may be regarded as clamped horizontally at one end, A, C or B, and bearing a load $\frac{1}{2}W$ at the free end E or F. In these conditions the directions of AE, CE coincide at E, and those of CF, BF at F. This equality in the deflexions under equal load implies that

$$AE=EC=CF=FB=\tfrac{1}{4}l.$$

Hence the depression of E, F below A, B and of C below E, F is

$$\begin{aligned} y&=\tfrac{1}{2}W(\tfrac{1}{4}l)^3/3Ak^2E \\ &=Wl^3/384Ak^2E, \end{aligned}$$

and the depression of C below AB is

$$2y=Wl^3/192Ak^2E,$$

or one-quarter of that obtained when the beam is supported freely at A, B.

The moment of the couple at A is $\frac{1}{2}W.AE=\frac{1}{8}Wl$, so that

$$\tfrac{1}{8}Wl=Ak^2E/\rho,$$

or

$$\rho=8Ak^2E/Wl$$

is the radius of curvature at A.

277. When the ends of a rod AB, of length l, are connected by a taut string under a tension T, the bending moment at a point P is

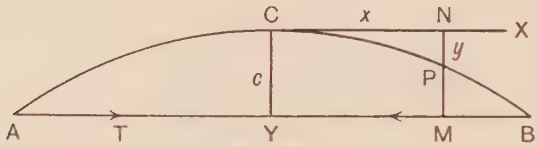

FIG. 162.

$PM\,.\,T$, where PM is perpendicular to AB. Hence in this case

$$Ak^2E/\rho=T\,.\,PM,$$

or

$$\rho\,.\,PM=Ak^2E/T=\text{const.},$$

and this relation between ρ and PM determines the form of the rod. At B, $PM=0$ and the rod has no curvature.

Take CX parallel to AB, or the tangent at the middle point of the rod, as the axis of x, so that $CN=x$, $PN=y$ determine the position of P. It is not appropriate to write

$$y=x^2/2\rho-bx^3$$

as for the loaded rod, because the curve ACB is clearly symmetrical about CY and the relation must be equally satisfied by (x, y) and $(-x, y)$. Hence for a first approximation the form

$$y=x^2/2\rho-bx^4$$

is suggested, where ρ is the radius of curvature at C.

At B the rod is straight, and therefore, following an earlier argument, $y_1+y_3=2y_2$, or

$$(x+h)^2/2\rho-b(x+h)^4+(x-h)^2/2\rho-b(x-h)^4=2(x^2/2\rho-bx^4),$$

or

$$(x^2+h^2)/2\rho-b(x^4+6x^2h^2+h^4)=x^2/2\rho-bx^4,$$

or

$$h^2\left(\frac{1}{2\rho}-6bx^2\right)-bh^4=0.$$

The second term is negligible when h is made very small, and the condition obtained at B, where $x=\frac{1}{2}l$, is

$$3bl^2\rho=1.$$

Hence the approximate form of the curve is given by the relation

$$y=\frac{x^2}{2\rho}\left(1-\frac{2x^2}{3l^2}\right).$$

At B, $x=\frac{1}{2}l$, $y=c=CY$, and therefore

$$c=\frac{l^2}{8\rho}\left(1-\frac{1}{6}\right)=\frac{5}{48}l^2/\rho.$$

Also the bending moment at C is

$$cT=Ak^2E/\rho,$$

and hence, on division by the previous equation,

$$T=9{\cdot}6\,Ak^2E/l^2.$$

The coefficient $9{\cdot}6=3{\cdot}1^2$ and is an approximation to π^2, which would be obtained by a less crude method.

How is this last result to be interpreted? When T is small there is no reason why the rod should be curved at all. It will remain straight under moderate tension. When T reaches this critical value, the equilibrium in the straight form becomes unstable, and any disturbance will make it assume the form here approximately determined. When T is increased beyond this limit, a flexible rod becomes still more curved and can be made to assume a variety of forms, each of which is known as an *elastica*. The very restricted limits of our approximation make it quite impossible to follow out these forms. It is necessary to stop at the very first stage. But even this gives the solution of such a practical problem as the stability of a vertical column under a load.

278. In the examples of bending considered the results have been presented as geometrical approximations only. They are, however, something more, in spite of their expressed limitations. To return to equation (6),

$$C=Ak^2E/\rho$$

shows that the bending moment C varies as Ak^2E and as the curvature ρ^{-1}. Thus the expression Ak^2E, which depends on the cross-section of the rod and on Young's modulus for the material of which it is composed, is a measure of the effective resistance of the rod to bending under a given moment. It may therefore be called the flexural stiffness of the rod and conveniently denoted by a single symbol K.

In the examples contemplated the rod is nearly straight and

horizontal. The axis of x will be taken horizontal, and the axis of y vertically downwards. Then (§ 87)

$$\frac{1}{\rho}=\frac{d^2y}{dx^2}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{-\frac{3}{2}}=\frac{d^2y}{dx^2}\cdot\cos^3\psi\,,$$

where $\tan\psi=dy/dx$ or ψ is the inclination of the tangent to the rod to the axis of x. So long as this inclination is small, as supposed, it is fair to take $\cos\psi=1$, and thus, by a combination of the above equations,

$$K\frac{d^2y}{dx^2}=M,$$

where M (in the place of C) represents the bending moment at a distance x along the rod.

The signs of the two sides of this equation must be consistent. When the rod is curved downwards, or convex upwards, the angle ψ and dy/dx increase with x, and d^2y/dx^2, the rate of increase of dy/dx, is positive.

Hence the bending moment M must be taken to be positive also when the sense of the stress is such as to cause compression in the lower surface, and tension in the upper surface, of the rod.

Let the form of the rod be assumed to be

$$y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4,$$

where y is the depression below the axis of x at a horizontal distance x from the origin. If the origin O is a point on the rod, $a_0=0$. The slope of the rod, which must be slight, is given at any point by

$$\tan\psi=\frac{dy}{dx}=a_1+2a_2x+3a_3x^2+4a_4x^3,$$

and therefore $$M=K\frac{d^2y}{dx^2}=K(2a_2+6a_3x+12a_4x^2).$$

The form assumed for the rod is therefore suitable over any range in which the rod is uniform, or K is constant, and the bending moment is a continuous quadratic function. Now this is the case between distinct points where finite forces are applied to the rod, provided the load, including the weight of the rod itself, is uniformly distributed over the interval. When the bending moment M is defined, the coefficients a_2, a_3, a_4 are clearly determined. The coefficients a_0, a_1 depend on the choice of origin and other circumstances of the problem. Thus $a_1=\tan\psi$ when $x=0$. At a free end of the rod, $M=0$ and the curvature vanishes.

279. *Ex.* A rod, of length l and weight w per unit length, is held at one end and supports a weight W attached to the other. To find the form of the rod when both ends are at the same level.

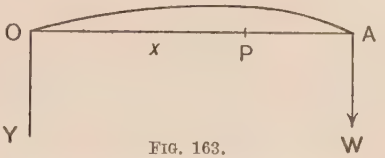

FIG. 163.

The origin O is taken at the fixed end, so that $y=0$ both when $x=0$ and when $x=l$. The form of the rod may therefore be taken to be

$$y=x(l-x)(a+bx+cx^2)$$
$$=alx+(bl-a)x^2+(cl-b)x^3-cx^4.$$

Then the bending moment must be

$$M=K\frac{d^2y}{dx^2}=K\{2(bl-a)+6(cl-b)x-12cx^2\}.$$

But if $OP=x$, the bending moment due to the weight W and the length $PA=l-x$ of the rod is (cf. § 189)

$$M=W(l-x)+\tfrac{1}{2}w(l-x)^2.$$

Hence, by equating the coefficients in the two expressions of M,

$$2K(bl-a)=l(W+\tfrac{1}{2}wl),\quad 6K(cl-b)=-W-wl,\ -12Kc=\tfrac{1}{2}w,$$

and therefore

$$-Kc=\tfrac{1}{24}w,\quad Kb=\tfrac{1}{6}(W+\tfrac{3}{4}wl),\quad -Ka=l(\tfrac{1}{3}W+\tfrac{1}{8}wl).$$

The coefficients being thus determined,

$$Ky=-x(l-x)\{l(\tfrac{1}{3}W+\tfrac{1}{8}wl)-(\tfrac{1}{6}W+\tfrac{1}{8}wl)x+\tfrac{1}{24}wx^2\},$$

or
$$24Ky=-x(l-x)\{(8W+3wl)l-(4W+3wl)x+wx^2\},$$

which gives the elevation, $-y$, of the rod at any intermediate point above its ends.

280. When a rod is supported at any number of distinct points in a nearly horizontal position, the conditions in each section or span must be considered separately because the shearing stress changes discontinuously at each point of support. It is, however, possible to find a relation between the bending moments at any three adjacent points of support.

Let A, O, B be three such points in order, with $OA=a$, $OB=b$. Per unit length the uniform load is w_1 on OA, w_2 on OB. The point O will be taken as the origin, with the axis of y vertically downwards and the horizontal axis of x in the direction of A, in the first instance. Let c_1, c_2 be the (small) vertical depths of A and B

below O. Immediately on either side of O the shearing stresses are S_2, S_2' so that S_2 is the vertical force applying to the span OA,

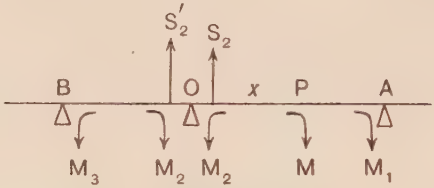

FIG. 164.

S_2' the vertical force applying to the span OB, and the joint corresponding reactions, $S_2+S'_2$, constitute the pressure on the support at O. The bending moments at A, O, B are M_1, M_2, M_3, and, if $OP=x$, the moment at P is M.

The equilibrium of the part OP of the rod requires that

$$M=M_2+\tfrac{1}{2}w_1x^2-S_2x,$$

and, when P coincides with A,

$$M_1=M_2+\tfrac{1}{2}w_1a^2-S_2a,$$

so that $$aM=aM_2-(M_2-M_1+\tfrac{1}{2}w_1a^2)x+\tfrac{1}{2}w_1ax^2.$$

But if, on this span,

$$y=a_1x+a_2x^2+a_3x^3+a_4x^4,$$

$$\frac{dy}{dx}=a_1+2a_2x+3a_3x^2+4a_4x^3=\tan\psi,$$

$$K\frac{d^2y}{dx^2}=K(2a_2+6a_3x+12a_4x^2)=M.$$

Hence, by comparison of the coefficients in M,

$$Ka_2=\tfrac{1}{2}M_2,\quad Ka_3=-(M_2-M_1+\tfrac{1}{2}w_1a^2)/6a,\quad Ka_4=\tfrac{1}{24}w_1.$$

Now $y=c_1$ when $x=a$, or

$$c_1=a_1a+a_2a^2+a_3a^3+a_4a^4,$$

so that $$Kc_1/a=Ka_1+\tfrac{1}{2}aM_2-\tfrac{1}{6}(M_2-M_1+\tfrac{1}{2}w_1a^2)a+\tfrac{1}{24}w_1a^3$$
$$=Ka_1+\tfrac{1}{3}aM_2+\tfrac{1}{6}aM_1-\tfrac{1}{24}w_1a^3.$$

Similarly the conditions of the span OB will give

$$Kc_2/b=-Ka_1+\tfrac{1}{3}bM_2+\tfrac{1}{6}bM_3-\tfrac{1}{24}w_2b^3,$$

because $a_1=\tan\psi$ at O, and the slope of the rod is reversed to the right and left of this point. By addition,

$$24K\left(\frac{c_1}{a}+\frac{c_2}{b}\right)=8(a+b)M_2+4(aM_1+bM_3)-(w_1a^3+w_2b^3),$$

and this result is known as the equation of *Three Moments*.

281. When a rod is unloaded except at some definite point, the bending moment is a linear function of the distance along the rod,

and after the consideration of a quadratic function which results from uniform loading it is unnecessary to return to the problems treated geometrically in §§ 273-276, though the reader may find it profitable to make a fresh discussion of these examples for himself. But the importance of the problem treated in § 277 will justify a further examination of it.

The fundamental relation which determines the form of the rod is

$$K\rho^{-1}=T\,.\,PM,$$

where $K=Ak^2E$ is the flexural stiffness and T is the tension applied.

Also (Fig. 162), if $PM=\eta$,

$$PM=\eta=c-y;$$

and, with an approximate expression for the curvature,

$$\frac{1}{\rho}=\frac{d^2y}{dx^2}=-\frac{d^2\eta}{dx^2}.$$

Hence

$$K\frac{d^2\eta}{dx^2}=-T\eta,$$

an equation of which the form of solution is familiar. In fact, since $\eta=c$ and $d\eta/dx=0$ when $x=0$, the appropriate solution is

$$\eta=c\cos\left(\frac{T}{K}\right)^{\frac{1}{2}}x.$$

In the case of a single span of slight curvature, $\eta=0$ when $x=\frac{1}{2}l$ approximately, where l is the length of the rod. Then

$$\frac{1}{2}\pi=\left(\frac{T}{K}\right)^{\frac{1}{2}}\cdot\frac{1}{2}l,\quad T=\pi^2K/l^2;$$

and this is the result previously suggested.

But the solution is periodic, and can be interpreted more generally. Instead of a single span the rod may assume the form of several spans, alternately on either side of the line of the tension T. A single span, of length l, represents a half-period, and a rod of length L may thus be deformed into a sine curve of r single spans, where

$$L=rl,\quad l=\pi\left(\frac{K}{T}\right)^{\frac{1}{2}}.$$

This result is independent of the greatest deflexion of the span, which must, however, be small to justify the argument.

282. When the deflexion of the rod is considerable, the approximate method employed above becomes invalid, and another must

be substituted. The fundamental expression for the curvature of the rod may be written

$$K\frac{d\psi}{ds}=T(c-y),$$

where (Fig. 162) ψ is the inclination of the tangent at P to the line CX. Hence

$$K\frac{d^2\psi}{ds^2}=-T\frac{dy}{ds}$$
$$=-T\,.\,\text{Lim.}\frac{\delta y}{\delta s}=-T\sin\psi$$

(cf. § 79). Now this is analogous to the equation for a pendulum, which may be written (§ 234) in the form

$$L\frac{d^2\theta}{dt^2}=-g\sin\theta.$$

The equation of energy for the pendulum is (§ 111 or § 133)

$$\frac{1}{2}L\left(\frac{d\theta}{dt}\right)^2=g\cos\theta+C,$$

and the corresponding equation for the rod is

$$\frac{1}{2}K\left(\frac{d\psi}{ds}\right)^2=T\cos\psi+C;$$

or, if the constant $C=-T\cos\gamma$,

$$T^2(c-y)^2=K^2\rho^{-2}=2KT(\cos\psi-\cos\gamma).$$

Beyond this equation, connecting the curvature ρ^{-1} of the rod with ψ, the corresponding inclination of the rod to the line of the force T, it is not possible to go without passing the present limitations. General deductions must therefore be based on this equation, and on the analogy between the form of the rod and the motion of the pendulum.

In this analogy the time t is represented by the length s measured along the rod, which should not be conceived necessarily as limited in either direction. The angle ψ corresponds with the inclination θ of the pendulum to the vertical. It follows that the angle γ corresponds with the amplitude of the oscillation and the curvature of the rod with the angular velocity of the pendulum. Now the motion of the pendulum is one of two possible types, (1) oscillating, with $\gamma<\pi$ real, or (2) circulating, with $C>T$. In the latter case, the angular velocity never vanishes. Hence in the analogy of the rod, the curvature of the rod never vanishes. But when the rod crosses the line of the tension T, $y=c$ and the curvature does vanish. Hence when the rod is constrained by a string attached

to two of its points, the analogy to its form must be sought in the oscillating pendulum. There is, indeed, a form of the *elastica* which does not cross the line of the tension T, and which corresponds with the circulating pendulum, but this is of theoretical interest only, since it satisfies the fundamental equation but not the physical conditions here assumed.

Thus the practical analogy is to be found in the oscillating pendulum, with γ real. At the points where the rod crosses the line of the tension, the curvature vanishes and $\psi = \gamma$. The form of the rod is periodic, and in a single span its inclination to the line of the tension changes from 0 at the centre to γ at the ends. When the deflexion is small, it has been seen that the form approaches that of a sine curve. As γ increases, this elementary shape is deformed, but is tolerably preserved till $\gamma = 90°$, when the line of tension is crossed at right angles. Beyond this stage, $\gamma > 90°$ and the arc CB (Fig. 162) turns through more than a right angle, so that the base AB becomes narrower than the greatest width of the span. Still further in this sequence the points A and B first coincide and then cross one another, finally receding to an indefinite distance, with a single loop of the rod between them. For a series of figures showing this sequence of changes, which apart from theory are easily tested by experience, the reader is referred to Thomson and Tait, *Natural Philosophy*, Part ii. p. 148.

EXAMPLES.

1. To a light elastic string n equal masses, each of weight w/n, are attached at equal distances from one another and from the ends of the string. The length of the string is l and the effective modulus λ. If the string is hung up by one end and supports an additional weight W at the other, show that the extension of the string beyond its natural length is $(W + \frac{1}{2}w)l/\lambda$.

2. In the same circumstances, prove that the potential energy of the string in the position of equilibrium is

$$\frac{l}{2\lambda}\left[W^2 + Ww + \frac{2n+1}{6n}w^2\right].$$

3. When an elastic string, loaded in any way, is suspended by one end, prove that the work done in the vertical descent from the unstrained position to the position of equilibrium is double the energy of the string in the latter position.

4. An elastic string, at first under a given tension T, is stretched until its length is further increased by a given amount a. Show that

the total energy stored in the string in the final state will be least if the extension beyond the natural length is already a in the first state.

5. A weight W is placed on a rough inclined plane, the inclination being α and the coefficient of friction μ. It is attached to a fixed point in the plane by an elastic string of length l with the modulus λ. Show that the weight can rest on the line of greatest slope through the fixed point over a range $2\mu W.l\cos\alpha/\lambda$ if $\mu < \tan\alpha$, or at any point of a range greater than this by $2l$ if $\mu \geqq \tan\alpha$.

6. Two fixed points on the same horizontal level, $2a$ apart, are connected by a light string of which the natural length is $2a$ and the modulus λ. When a weight W is attached at the middle point of the string, show that this point is drawn down a distance $a(W/\lambda)^{\frac{1}{3}}$ approximately.

7. A light elastic string, of natural length l and modulus λ, supports a weight W in the vertical position of equilibrium. The system is then set in motion so that the weight describes a horizontal circle with the string inclined at the angle α to the vertical. Prove that the total increase in the energy of the system is

$$Wl\left\{\frac{(1-\cos\alpha)(1+3\cos\alpha)}{2\cos\alpha}+\frac{W}{\lambda}\tan^2\alpha\right\}.$$

8. A weight is suspended by a light elastic string. Prove that the period of vertical oscillation is the same as the period of a simple pendulum equal in length to the extension of the string under the same static load.

A weight of one kg. hangs by a steel wire 3 metres long and 0·5 mm. in diameter. Find the frequency of vertical oscillations, being given that Young's modulus for steel is 2×10^{12} C.G.S. units.

9. Two parallel bars, which may be considered rigid, are connected by three equal rods of the same elastic material at right angles to the bars. Equal and opposite forces applied to the bars on the line of one of the rods produce a longitudinal strain α in this rod and strains β and γ in the other rods. Show that

$$\alpha(\beta+\gamma)=\beta^2+\gamma^2.$$

10. A rhombus $ABCD$ is formed of four equal similar rods freely jointed, and the opposite pairs of hinges are joined by two rods AC, BD of the same gauge and material as the others. If the hinges at A and C are drawn apart with equal opposite forces P, prove that the tension in the rod AC, those in the sides of the rhombus and the thrust in the rod BD are respectively

$$Q(1+2\cos^3\theta),\quad Q\sin^2\theta,\quad Q\sin\theta\sin2\theta,$$

where 2θ is the angle ABC and $Q(1+2\sin^3\theta+2\cos^3\theta)=P$.

11. A uniform rigid beam, of weight W, is placed on three equal, similar vertical props fixed in the ground at A, B, C. Prove that the pressures on the three props are respectively

$$Q(AB \,.\, NB + AC \,.\, NC), \quad Q(BC \,.\, NC + BA \,.\, NA),$$
$$Q(CA \,.\, NA + CB \,.\, NB),$$

with due regard to the sign of the segments, where

$$Q(AB^2 + BC^2 + CA^2) = W,$$

and N is the projection of the centre of gravity of the beam on the line ABC.

12. In the same circumstances, prove that the small tilt of the beam to the horizon is $3Qh \,.\, NG/ES$, where h and S are the height and sectional area of the three supports, E is Young's modulus for their material, and G is the centre of gravity of three equal particles at A, B, C.

13. A rectangular table, with equal elastic legs at the corners, stands on a rigid floor. The top $ABCD$ of weight W is also rigid, and a weight w is placed at a point P on it. If AP produced meets the diagonal BD in Q, prove that the pressure on the leg at A is

$$\tfrac{1}{4}(W + w) + \tfrac{1}{2}w \,.\, \frac{PQ}{AQ}.$$

14. A sheet of metal is subjected to uniform like stresses T_1, T_2 crossed at right angles. If ϕ is the angle between the resultant stress across any line and the normal to the line, show that the greatest value of ϕ is given by

$$\sin \phi = \frac{T_1 - T_2}{T_1 + T_2}.$$

15. A plate is under uniform tension T in one direction and uniform pressure P in the perpendicular direction. Show that there is no elongation in the line which makes the angle θ with the direction of pressure, where

$$\tan^2 \theta = (P + \sigma T)/(T + \sigma P),$$

σ being Poisson's ratio.

16. A square slab of material is placed between two parallel plates which prevent expansion, and is subjected to a uniform pressure on two opposite faces. Show that the ratio of stress to strain in the direction of the pressure is $E(1 - \sigma^2)^{-1}$, where E is Young's modulus and σ is Poisson's ratio for the material of the slab.

17. A cube is subject to normal stresses on its six faces, such that the resulting strain is a simple elongation parallel to one edge. Show that the ratio of stress to strain in this direction is $k + \frac{4}{3}n$, where k is the bulk modulus and n is the rigidity.

18. Show that a longitudinal extension in any direction is equivalent to a uniform cubical dilatation together with two equal shears. Find the amount and situation of the shears.

19. A hollow cylinder of length l, with the external and internal radii given, has one plane end fixed while a torque N is applied to the other. Show that the angle through which this end is turned relatively to the fixed end is Nl/nI, where n is the rigidity and I is the moment of inertia of a plane section about the axis of the cylinder. Show also that the potential energy of the cylinder in equilibrium is $N^2l/2nI$.

20. A solid shaft, d inches in diameter, makes f revolutions per minute and transmits P horse-power. Show that the greatest shearing stress q (in pounds' weight per square inch) is given by

$$q \cdot fd^3 = 321{,}000\ P.$$

21. A solid cylindrical shaft of length l and radius a, rotating on its axis with frequency f transmits the power P. Show that, with consistent units,

$$lP = (\pi a^2)^2 f n \phi,$$

where n is the modulus of rigidity and ϕ is the angular twist of one end relative to the other.

Find the total twist in an iron rod ($n = 5 \times 10^{11}$ C.G.S. units) a metre long, 2 cm. in diameter, transmitting 1 kilowatt when making 200 revolutions per minute.

22. A uniform rod of length l is fixed horizontally at one end, with the other end free. Prove that the droop y at a horizontal distance x from the fixed end is

$$y = \frac{w}{24\,\mathrm{K}} x^2(6l^2 - 4lx + x^2),$$

where w is the weight per unit length and K is the measure of stiffness.

23. A uniform rod of length $2a$ is freely supported at the two ends on the same horizontal level. Prove that the droop y below the line of support at a horizontal distance x from the point midway between the supports is

$$y = \frac{w}{24\,\mathrm{K}} (a^2 - x^2)(5a^2 - x^2),$$

where w is the weight per unit length and K is the stiffness.

24. A uniform rod of length $2a$ is fixed horizontally at both ends on the same level. Prove that the droop y below the fixed ends at a horizontal distance x from the point midway between them is

$$y = \frac{w}{24\,\mathrm{K}} (a^2 - x^2)^2,$$

where w is the weight per unit length and K is the stiffness.

25. A uniform beam, of weight W, rests horizontally on three supports, two at the ends and one at distances a and b from the two ends. Prove that the pressures on the supports are in order

$$\frac{3a^2 + ab - b^2}{8a(a+b)} W, \quad \frac{a^2 + 3ab + b^2}{8ab} W, \quad \frac{3b^2 + ab - a^2}{8b(a+b)} W.$$

ANSWERS TO EXAMPLES

CHAPTER I

(p. 31) **1.** 36 miles S. **6.** Two.

7. Four (five if lines are parallel).

(p. 32) **16.** Ex. 11, 12 correspond with § 16.

18. (i) $-\frac{g}{b}$; (ii) $\frac{c^3}{2b^2}$; (iii) $\frac{3}{2}a$ for branch touching either axis.

CHAPTER II

(p. 57) **2.** $\frac{1}{2}v(t+T)$. **3.** $\frac{1}{4}$ hr.; 5 hrs. **4.** After 20 secs. in 600 ft.

7. (*a*) $(f_2-f_1)(f_1T+2v_1)>0$,

(*b*) $(f_2-f_1)(f_1T+v_1-v_2)>0$, and

$(v_1+v_2)^2=(2v_1+f_1T)(2v_2-f_2T)$.

8. $\frac{v}{g}\left[\frac{1}{1+f}+\frac{1}{\sqrt{(1-f^2)}}\right]$; $f=\frac{1}{\sqrt{2}}$.

(p. 58) **9.** 227 ft. **11.** (*a*) $\frac{1}{2}n^2a$; (*b*) $2n^2a/\pi$.

12. $\frac{176}{9}$ radians per sec.; $\frac{220}{9}$ ft./sec.; $\frac{38720}{81}$ ft./sec.^{2}.

(p. 59) **18.** $\dot{x}=\frac{1}{2}at^{-\frac{1}{2}}$; $\ddot{x}=-\frac{1}{4}at^{-\frac{3}{2}}$.

19. $nV\sin nt\,(4\cos nt-1)$; $\frac{V}{n}\sin nt\,(1-\cos nt)$; $t=\frac{2r\pi}{3n}$;

$\cos nt=\frac{1}{4}$.

CHAPTER III

(p. 88) **1.** 33·8 ft./sec.^{2} towards W.N.W.

(p. 89) **10.** 60 yds. **16.** 281 miles/hr. **17.** 49·7 cm.

(p. 90) **18.** 1; -1; -3.

23. (1) $(1+\sin^2 x)^{\frac{3}{2}}\sec x$; (2) $(2+\cos 2t)^2/\sqrt{2}\sin t$;

(3) $\left(\frac{a^2+b^2+2ab\sin\theta}{a^2+2b^2+3ab\sin\theta}\right)^{\frac{3}{2}}$.

25. $(u^2+n^2r^2)^{\frac{3}{2}}/n\,(2u^2+n^2r^2)$.

CHAPTER IV

(p. 117) **1.** $\frac{M-m}{2m}\,.\,g$ upward.

2. $\frac{m_2-m_1}{M+m_1+m_2}\,.\,g\,;\quad \frac{M+2m_2}{M+m_1+m_2}\,.\,m_1g\,;\quad \frac{M+2m_1}{M+m_1+m_2}\,.\,m_2g.$

3. $4m_1m_2/3(m_1+m_2)$.

(p. 118) **11.** $\mu_2 > \mu_1$ if M_2 is above M_1 ;

$$\text{accn., } g\left[\sin\alpha - \frac{\mu_1m_1+\mu_2m_2}{m_1+m_2}\cos\alpha\right],$$

$$\text{tension, } g\cos\alpha\,.\,\frac{m_1m_2}{m_1+m_2}\,(\mu_2-\mu_1).$$

12. $(99{\cdot}0968 + 0{\cdot}5252\sin^2\phi)$ cm. ; 61° 11′.

13. 331 ; 0·32 in.

(p. 120) **22.** 13·05 lbs. wt.

CHAPTER V

(p. 151) **6.** $\tan D = \dfrac{-m_2(1+e)}{2\{(m_1+m_2)(em_2-m_1)\}^{\frac{1}{2}}}\tan 2\psi,$

where $\tan\psi = \left(\dfrac{m_1+m_2}{em_2-m_1}\right)^{\frac{1}{2}}\tan\theta.$

8. $2{\cdot}5R$ above lowest point of circle. **10.** $\frac{1}{107}$ sec.

11. (*a*) $P^2/2mR$, (*b*) $P^2/2MR$; R = resistance.

(p. 152) **14.** $\sqrt{(\frac{2}{5}gl)}$. **15.** $a=(V_1+V_2)V_2/2g$, $b=(V_1+V_2)V_1/2g$.

16. 11·6 secs. losing. **18.** 91 H.P. ; 5·93 in.

(p. 153) **19.** 1·04 sec. **21.** 157·5.

22. 0·5887 ; 7·7 knots. **23.** $M^{-1}L^3T^{-2}$.

CHAPTER VI

(p. 185) **11.** $\dfrac{bc(b-c)^2}{4a^3}$, $\dfrac{c(b^2+bc+2c^2)}{4a^2}$, $\dfrac{b(2b^2+bc+c^2)}{4a^2}$.

(p. 186) **18.** $2b+\dfrac{2aA}{2A+B}+\dfrac{2cC}{2C+B}$.

CHAPTER VII

(p. 223) **1.** Line of action makes angle 175·4° with AB, and perpendicular from A is 635.

(p. 225) **17.** Bar divides vertical sides in ratio $W+w : W+5w$ downward ; $\sqrt{3}(W+3w)$.

(p. 227) **25.** $2\sqrt{5}$ lbs. wt.

CHAPTER VIII

(p. 264) **3.** $M \,.\, GB/AB$ at A, $M \,.\, AG/AB$ at B.

(p. 265) **7.** $$\frac{I_1 \cos(\beta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)} + \frac{I_2 \cos(\gamma-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)} + \frac{I_3 \cos(\alpha-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}.$$

9. $h + 4a^2/h$ (h = distance of chord from centre, a = radius).

(p. 266) **18.** $2\pi \left\{ \dfrac{3a^2-2c^2}{3g\sqrt{(a^2-c^2)}} \right\}^{\frac{1}{2}}$.

(p. 267) **23.** $gh^2/(h^2+4\pi^2k^2)$.

CHAPTER IX

(p. 300) **8.** 18·2 per sec.

(p. 302) **21.** 3·5°.

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